A

COLLECTION OF PROBLEMS

IN ILLUSTRATION OF THE PRINCIPLES

OF

ELEMENTARY MECHANICS.

BY

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PREFACE.

In the composition of this work, my object has been to arrange in a systematic form a Collection of Mechanical Problems for the use of the higher Schools and Various Colleges of this Country, and especially for the service of those members of the University of Cambridge who are engaged in the study of the Elementary course of Mechanics, with which Candidates for Honours are expected to be familiar in the first three Days of the Examination for the Mathematical Tripos. In the Schedule of the subjects, as fixed by the Grace of the Senate for the regulation of the Examination of the Candidates during these three days, the range of Mechanical reading is limited to the following branches:—

"The elementary parts of Statics, treated without the Differential Calculus; namely, the Composition and Resolution of Forces acting in one plane on a point, the Mechanical Powers, and the Properties of the Centre of Gravity."

"The Elementary parts of Dynamics, treated without the Differential Calculus; namely, the Doctrine of Uniform and Uniformly accelerated Motion, of falling Bodies, Projectiles, Collision, and Cycloidal Oscillations."

The whole number of propositions in Elementary Mechanics, as implied in these extracts from the Schedule, are certainly far from numerous: indeed any sensible student, acquainted with the rudiments of Geometry, Algebra, and Trigonometry, might, in a very short time, acquire so rational a comprehension of the truth of the demonstrations, as to be able to pass a

satisfactory examination in the mere book learning of the subject. The most valuable effect however of the study of science is to awaken originality or at least activity of thought, and to prepare the student for a state of intellect beyond mere logical assent to a number of tolerably easy propositions. In order that his mind may be aroused from the unproductive torpidity of mere acquiescense in prescribed demonstrations of standard theorems, it is necessary that he should exercise his invention in the solution of problems by the application of his primary principles and theorems. As the best method of acquiring a power of independent thought, I would strongly urge him to be in the habit of concentrating his attention on the kindred properties of each fundamental idea, and to abandon the too common practice of the indiscriminate solution of problems connected together by no principle of affinity. In order to facilitate this process of study, I have written this and other similar works. By the classification of numerous problems in distinct groups, I have endeavoured to induce the student to expend an adequate amount of thought at one time in the exclusive consideration of the phenomena of each important theorem, regarded as a characteristic type of its corresponding group, hoping to convey to his mind, through the medium of an enlarged comparison of associated properties, an abiding and energetic conception of each grand principle of the Science.

For far the greater number of the problems, which are contained in this Volume, I am indebted to the printed Examination papers, proposed by the University Examiners for the Mathematical Tripos and by the Examiners in the various Colleges. My reason for so largely availing myself of these sources is twofold; first, because it is impossible for any one man to invent a large number of problems equal in value to the best of those contributed by many; and, secondly, because, by bringing together, as I have done, so many of the problems which have of late years been actually proposed to Candidates for the various University and College distinctions, I have prepared a

collection of problems which may certainly be regarded as fair specimens of the class of problems to which, in this branch of Mechanics, the University of Cambridge directs the attention of its younger members.

As the doctrine of Cycloidal Oscillations is especially mentioned in the Schedule of the subjects prescribed for the First Three Days of the Examination for the Mathematical Tripos, I have thought it desirable to publish, in an Appendix to this Volume, a new demonstration, by Mr R. L. Ellis of Trinity College, of the tautochronism of the Cycloid, which I think cannot fail to be interesting to many of those who are engaged in the study of Elementary Mechanics.

The utility of a work of this kind to a beginner in mathematical studies would be much impaired, were the answers to the various problems, the solutions of which are not given in full, very frequently erroneous. I have accordingly taken much pains to render the results accurate: I cannot however hope to have escaped the commission of sundry errors, and should feel much indebted to any one who would be so kind as to direct my attention to any mistakes, the rectification of which might be effected either in a new table of errata or ultimately in a second edition of the work, if called for by the public.

CAMBRIDGE,
September 24, 1858.

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Page 73, line 17,	Correction.
d^3	$4(a+b+c)^3 d^2$

STATICS.

CHAPTER I.

RESULTANT OF TWO FORCES ACTING ON A POINT.

SECT. 1. *Geometrical Method.*

IF two forces P, Q, act upon a point O, fig. (1), in the directions OA, OB, respectively, and if

$$OA : OB :: P : Q,$$

then OC, the diagonal of the parallelogram $OACB$, will coincide with the direction of the resultant R of the two forces, and

$$OA : OB : OC :: P : Q : R.$$

(1) Two forces of 6 and 8 lbs., acting on a point, have a resultant 10 lbs.: to find the angle between the two former forces.

Let O, fig. (1), be the point, OA, OB, the directions of the forces 6, 8, respectively: take $OA = 6$ units of length, and $OB = 8$ units of length: complete the parallelogram $OACB$, and join OC. Then OC must be equal to 10 units of length.

Since $6^2 + 8^2 = 10^2$, the square on OC must be equal to the square on OA together with that on AC: hence, by Euclid, the angle OAC, and therefore the angle AOB, that is, the angle between the two forces, must be a right angle.

(2) If the angle between two equal forces acting on a point be $120°$, what is their resultant?

Let CA, CB, fig. (2), represent the two forces: complete the parallelogram $CBDA$, and join CD: then CD will represent the resultant.

Now	$AD = BC = CA$;
hence	$\angle\, CDA = \angle\, DCA$:
but	$\angle\, CDA = \angle\, DCB$:
hence	$\angle\, DCA = \angle\, DCB$,

and therefore, since $\angle\, ACB = 120^\circ$, $\angle\, DCA = 60^\circ = \angle\, CDA$: but the sum of the angles of the triangle ACD is 180°: hence also $\angle\, CAD = 60^\circ$, and the triangle CAD is equilateral. Thus CD is equal to CA or CB and bisects the angle ACB, that is, the resultant is equal to either of the component forces and bisects the angle between them.

(3) Two forces act on a point at right angles to each other and the resultant is double of the less: shew that the angle which the resultant makes with the less is double the angle which it makes with the greater.

Let CA, CB, fig. (3), represent the two forces: complete the rectangle $BCAD$ and draw the diagonal CD, which will represent the resultant of CA, CB.

Bisect CD in O and join OA: then, since a circle can be described about the triangle CAD, having CD for its diameter, it is plain that OD, OA, which are radii, must be equal to each other: but $AD = OD$, by the hypothesis: hence the triangle AOD is equilateral and therefore equiangular.

Again, OA, OC, being radii, the triangle AOC is isosceles, and $\angle\, ACO = \angle\, CAO$.

The exterior angle AOD of the triangle ACO is equal to the angle ACO together with the angle CAO, and therefore to twice the angle ACO:

but	$\angle\, BCD = \angle\, ODA = \angle\, AOD$:
hence	$\angle\, BCD = 2\, \angle\, ACD$.

(4) At any point of a parabola forces are applied, represented in magnitude and direction by the tangent and normal at the point: to prove that, the forces being supposed to act both towards or both from the axis, the resultant will pass through the focus.

Let PT, PG, fig. (4), be the tangent and normal at any point P of the parabola. Then the force PT acting on P is equivalent to forces PS, ST, acting on P; and the force PG acting on P is equivalent to forces PS, SG, acting on P, or, since the line SG is equal to the line ST, to forces PS, TS, acting on P.

Hence the forces PT, PG, acting on P, are equivalent to the force $2PS$ acting on P in the direction PS.

(5) Two forces, which act at a point of a curve, are represented in magnitude and direction by chords passing through the point; if the position of one of the chords and the direction of the resultant be given, to determine the positions of the other chord.

Let BAC, fig. (5), be the curve; let the chord AP, given in position, represent the force; and let the indefinite line AS represent the given direction of the resultant of the force AP and the required force. Produce PA to T, making AT equal to AP. Through T draw the line TQ parallel to AS, cutting the curve in any point Q. Draw QR parallel to TA, cutting AS in R: join PR. Then QR, AP, are equal to one another, being each equal to AT; they have also been drawn parallel to each other. Hence, by Euclid, $AQ = PR$. Thus the chord AQ represents a force which, combined with the force AP, corresponds to a resultant AR.

If the line TQ, produced if necessary, cuts the curve in any other points Q', Q'', Q''', ...; there will be other chords AQ', AQ'', AQ''',... which satisfy the conditions of the problem.

(6) Two forces act along opposite sides of a quadrilateral in a circle, towards the same parts, and are respectively proportional to these sides: to prove that the resultant will pass through the intersection of the diagonals.

Let $ABCD$, fig. (6), be the quadrilateral, O being the intersection of the diagonals, and let the forces act along AB, DC, or along BA, CD. Produce AB, DC, to meet in T, and draw OE, OF, parallel to DC, AB, respectively.

The angle OBE is equal to $\angle AOB + \angle BAO$, and therefore to $\angle DOC + \angle BAC$, and therefore to $\angle DOC + \angle BDC$, or to $\angle DOC + \angle CDO$, and therefore to $\angle OCF$. Also $\angle BEO = \angle CFO$. Hence the triangles OBE, OCF, have the angles OBE, BEO, equal respectively to the angles OCF, CFO, and are therefore similar triangles. Again the triangles AOB, COD, are similar, since the angle BAO is equal to the angle CDO, and the angle AOB to the angle COD.

$$\begin{aligned} \text{Hence} \qquad TF : TE &:: OE : OF \\ &:: OB : OC \\ &:: AB : CD \\ &:: \text{force along } DC : \text{force along } AB. \end{aligned}$$

Hence the resultant of these two forces must pass through O.

(7) The resultant of two equal forces, applied at a given point, is represented in magnitude and direction by a given straight line drawn from the point: prove that the locus of the extremity of the straight line which represents either force is an indefinite perpendicular erected at the middle point of the given straight line.

(8) If ABC be a right-angled triangle and $ABDE$, $ACFG$, be the squares on the sides, constructed as in Euclid I. 47; prove that the resultant of forces represented by CD, BF, is parallel to a diagonal of the rectangle the sides of which are AE, AG.

(9) Having given a line representing in length the difference between the resultant and one of two equal forces, which act upon a point at right angles to each other, shew how the lines representing the forces may be found by a geometrical construction.

(10) Two forces act along opposite sides of a quadrilateral in a circle, towards opposite parts, and are respectively proportional to these sides; prove that their resultant passes through the intersection of the two other sides.

SECT. 2. *Trigonometrical Method.*

Let two forces P, Q, act upon a point, and let α be the angle between their directions: then, if R be their resultant, and if R's direction be inclined to the directions of P, Q, at angles ϕ, ψ, respectively,

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha,$$

$$\sin\phi = \frac{Q}{R}\sin\alpha, \qquad \sin\psi = \frac{P}{R}\sin\alpha.$$

The first and either of the last two of these formulæ determine the magnitude and the direction of the resultant.

If two forces X, Y, act at right angles to each other, and if their resultant R make angles λ, μ, with X, Y, respectively, then

$$R^2 = X^2 + Y^2,$$

$$\cos\lambda = \frac{X}{R}, \qquad \cos\mu = \frac{Y}{R}.$$

(1) If two forces, each equal to P, act upon a point, the angle between their directions being 60°, to find the magnitude of their resultant.

Let R denote their resultant: then

$$\begin{aligned} R^2 &= P^2 + P^2 + 2P.P.\cos 60^\circ \\ &= 3P^2, \\ R &= P\sqrt{3}. \end{aligned}$$

This question may be treated also as follows.

Let C, fig. (7), be the point at which the two forces act: draw Cx to bisect the angle between their directions, and draw yCy' at right angles to Cx. Then one of the forces is equal to

$P\cos 30^\circ$ along Cx and $P\sin 30^\circ$ along Cy, the other force being equal to $P\cos 30^\circ$ along Cx and $P\sin 30^\circ$ along Cy': the components along Cy, Cy', destroy one another, and there remains for the resultant a force along Cx equal to $2P\cos 30^\circ = P\sqrt{3}$..

(2) A cord PAQ is tied round a pin at the fixed point A, and its two ends are drawn in different directions by the forces P and Q: to determine the angle between these directions, supposing the pressure on the pin to be equal to $\frac{1}{2}(P+Q)$.

Let θ be the required angle: then, $\frac{1}{2}(P+Q)$ being by the hypothesis the resultant of the two forces,

$$\tfrac{1}{4}(P+Q)^2 = P^2 + Q^2 + 2PQ\cos\theta,$$

$$\cos\theta = \frac{2PQ - 3(P^2+Q^2)}{8PQ},$$

which determines the value of θ.

(3) A string ACP, fig. (8), with a weight P at the end of it, is attached to a point A in a wall AB: from a point B of the wall a thin rod BC projects horizontally, pushing out the string: to find the pressure of the string on the rod.

The end C of the rod is acted on by the tension P of the portion CA of the string, in the direction CA, and by the tension P of the portion CP of the string vertically downwards.

Let $\angle ACB = \alpha$: then the angle between the two equal forces P, P, acting at C, is equal to $\frac{1}{2}\pi + \alpha$: hence the pressure on the end C of the rod is equal to

$$2P\cos\frac{\pi + 2\alpha}{4}.$$

(4) A string is wrapped round a regular polygon, the tension of the string being given: to find the sum of the pressures on the angles of the polygon, and thence to determine the whole pressure on the circumference of a circle under like circumstances.

Let n be the number of the sides of the polygon; O, fig. (9), its centre; AB, BC, any two consecutive sides: join AO, BO,

CO. Let P be the tension of the string. Let R be the pressure on the angle B, being the resultant of the two equal forces P, P, acting along BA, BC. Then

$$R = 2P \cos ABO:$$

but
$$2 \angle ABO = \pi - \frac{2\pi}{n},$$

$$\angle ABO = \frac{\pi}{2} - \frac{\pi}{n}:$$

hence
$$R = 2P \sin \frac{\pi}{n}.$$

Consequently the sum of the pressures upon the n angles is equal to

$$2nP \sin \frac{\pi}{n}.$$

Let $n = \infty$: then the whole pressure upon the circumference of the circle, into which the polygonal periphery degenerates, is equal to

$$2\pi P . \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} = 2\pi P.$$

(5) The directions of two forces, equivalent to 3 lbs. and 4 lbs. respectively, are inclined to each other at an angle of 60°: to determine the magnitude and direction of their resultant.

$$R = \sqrt{(37)}, \quad \sin \phi = \frac{2\sqrt{3}}{\sqrt{(37)}}, \quad \sin \psi = \frac{3\sqrt{3}}{2\sqrt{(37)}}.$$

(6) If two equal forces are inclined to each other at an angle of 120°, shew that their resultant is equal to either of them.

(7) Two forces, acting on a particle, are inclined to each other at an angle of 30°, one force being equal to two pounds and the other to three pounds: to find the magnitude of their resultant.

The resultant is equal to

$$(13 + 6\sqrt{3})^{\frac{1}{2}}.$$

(8) If two forces, each equal to P, act at an angle of 135° to each other, shew that their resultant is equal to $(2-\sqrt{2})^{\frac{1}{2}}.P$.

(9) Two forces of 9 and 12 pounds act upon a point in directions at right angles to each other: to find the magnitude of their resultant.

The resultant is equivalent to a force of 15 pounds.

(10) Two forces act at right angles to each other, one of them being 4 pounds: to find the other when the resultant is 5 pounds.

The other is a force of 3 pounds.

(11) If two forces of 5 and 12 pounds act at the same point at right angles to each other, what single force will produce the same result?

The single force required = 13 pounds.

(12) If the magnitudes of two forces are 6 and 11, and the angle between their directions 30°, shew that the magnitude of their resultant is 16·47 nearly.

(13) Shew that, in the preceding question, the resultant makes with the force 6 an angle the sine of which is approximately ·3339, and with the force 11 an angle of which the sine is approximately ·1821.

(14) Two forces of 1 and 2 pounds act upon a point: to find the angle between the forces, supposing the resultant to be $\sqrt{7}$.

The required angle is 60°.

(15) Apply two pressures of P pounds and $P\sqrt{3}$ pounds so as to produce the same effect as a pressure of $2P$ pounds applied in a given direction.

The two forces P and $P\sqrt{3}$ must be at right angles to each other, their inclinations to the force $2P$ being 60° and 30° respectively.

(16) Shew that, if θ be the angle between two forces of given magnitude, their resultant is greatest when $\theta = 0$, least when $\theta = \pi$, and continually decreases as θ increases from 0 to π.

(17) Which will be the more effective, two men pulling with a single rope, the strength of each being $2P$, or two men pulling with two ropes at an angle of 90°, the strength of each being $3P$?

The men in the latter case will be more effective.

(18) Shew that the resultant of two forces P and Q, inclined to each other at an angle α, is equal in magnitude to that of two other forces

$$(P+Q)\cos\frac{\alpha}{2}, \quad (P-Q)\sin\frac{\alpha}{2},$$

making right angles with each other.

(19) Two forces P and nP act on a point and make a constant angle α with each other: to determine whether their resultant increases or decreases as n increases from zero.

If α be acute, the resultant constantly increases: if α be obtuse, the resultant decreases until $n = -\cos\alpha$, and afterwards increases, $P\sin\alpha$ being its least value.

(20) To find the angle between two forces P and Q when their resultant is a mean proportional between its greatest and least value.

If P be not less than Q, the required angle is equal to

$$\cos^{-1}\left(-\frac{Q}{P}\right).$$

(21) Two equal forces act on a particle, first at an angle of 45° to each other, and next at an angle of 90°: to compare the magnitudes of the resultants in the two cases.

The ratio of the former to the latter resultant is equal to

$$\left(1+\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}.$$

(22) Two equal forces P, P, act on a point, first, at an angle α, and then at an angle $\frac{1}{2}\pi - \alpha$: to compare the resultants in the two cases.

The required ratio is equal to

$$\frac{\sqrt{2}}{1 + \tan\dfrac{\alpha}{2}}.$$

(23) If the resultant of two equal forces P, P, acting on a point at an angle α, be n times as great as if the angle were 2α, to find the value of α.

The value of α is given by the relation

$$\cos\frac{\alpha}{2} = \frac{1 + (8n^2 + 1)^{\frac{1}{2}}}{4n}.$$

(24) Two equal forces are inclined to each other at such an angle that, if the direction of one of them were reversed, the resultant would be diminished in the ratio of $\sqrt{3}$ to 1: to find this angle of inclination.

The required angle is 60°.

(25) Two equal forces acting, at a certain inclination to each other, at a point, have a certain resultant: also, if the direction of one of the forces be reversed and its magnitude be doubled, the resultant is of the same magnitude as before: to determine the angle at which the forces are inclined.

The two original forces are inclined to each other at an angle of sixty degrees.

(26) The resultant of two forces, which act upon a point in directions perpendicular to each other and are in the ratio of 3 to 4, is equivalent to a weight of 60 pounds: to determine the forces.

The required forces are equivalent to weights of 36 and 48 pounds.

(27) The resultant of two forces acting at right angles to each other is 13 pounds: when each component is increased by

3 pounds, the resultant is equal to the sum of the two original components; to find the forces.

The required components are forces of 5 pounds and 12 pounds.

(28) Two equal weights P, P, are fixed to the ends of a string which passes over three tacks A, B, C, forming an equilateral triangle, A and C being in a horizontal line below B: to find the pressures on the tacks.

The pressure upon B is equal to $P\sqrt{3}$ and the pressure upon each of the tacks A and C is equal to

$$\frac{P}{\sqrt{2}}(\sqrt{3}-1).$$

(29) Three pegs A, B, C, are stuck in a wall in the angles of an equilateral triangle, A being the highest and BC being horizontal: a string, the length of which is equal to four times a side of the triangle is hung over them, and its two ends attached to a weight W: to find the pressure on each peg.

The pressure on A is equal to W, and the pressure on either B or C is equal to $\frac{W}{\sqrt{3}}$.

(30) A, B, C, fig. (10), are three fixed tacks: BC is horizontal and AB is inclined at an angle α to the horizon: a string $PABCP$ hangs over the tacks, sustaining equal weights P, P, at its ends: to find the pressures on the three tacks.

The pressures on A, B, C, are respectively equal to

$$2P\cos\frac{\pi+2\alpha}{4},\quad 2P\sin\frac{\alpha}{2},\quad P\sqrt{2}.$$

CHAPTER II.

EQUILIBRIUM OF THREE FORCES ACTING ON A POINT.

SECT. 1. *Triangle of forces.*

IF three forces, the magnitudes of which are proportional to the sides BC, CA, AB, of a triangle ABC, and of which the directions are either parallel to these sides or respectively inclined to them at equal angles, act upon a point, they will produce equilibrium, and conversely.

(1) A FORCE supports a weight on an inclined plane: supposing the force to act parallel to the plane and to be equal to the pressure on the plane, to find the inclination of the plane to the horizon.

Let P, fig. (11), be the force, R the reaction of the plane, and W the weight: from E, the intersection of W's direction with the horizontal line through any point A of the plane, draw EF at right angles to the plane. Then, since P, W, R, produce equilibrium, they must be proportional to the sides FO, OE, EF, respectively, of the triangle OEF. But $P=R$ by supposition: hence $FO=EF$, $\angle EOF=\angle FEO$, and therefore $\angle AOE$, and consequently $\angle OAE$, is equal to half a right angle.

(2) Can three forces in the ratio 5 : 3 : 2, acting on a point in any manner, keep it at rest? or forces in the ratio 10 : 6 : 3?

If three forces acting on a point produce equilibrium, they must generally be proportional to the three sides of a triangle. Now, when two sides of a triangle become together equal to the third, the triangle degenerates into a straight line, which may accordingly be regarded as a limiting state of a triangle: hence, in the former case, equilibrium is possible, the two forces 2 and 3 acting in a straight line oppositely to the force 5.

In the latter case equilibrium is impossible, because 10 is greater than $6 + 3$, and therefore the three forces cannot be represented by the sides of a triangle.

(3) Three forces, represented in magnitude and direction by the sides of a triangle, act on a point: if the greatest of the forces be to the least as 5 to 3, and the triangle be right-angled, to find their ratios to the other force.

Let the three forces be represented by $5P$, $3P$, nP: then, since the triangle is right-angled, and since its sides are proportional to the forces,

$$(5P)^2 = (3P)^2 + (nP)^2,$$

$$25 = 9 + n^2, \quad n^2 = 16, \quad n = 4.$$

Thus the three forces are in the proportion

$$5 : 3 : 4.$$

(4) Three forces P, Q, R, acting upon a point and keeping it at rest, are represented by lines drawn from that point: if the line which represents P be given in magnitude and direction, and that which represents Q be given in magnitude only, to find the locus of the extremity of the line which represents R.

Let O, fig. (12), be the point on which the three forces act: let AO represent the force P, CA the force Q: complete the triangle CAO: then, since P, Q, R, are in equilibrium, OC must represent the force R.

Since the line representing P is given in magnitude and direction, A is a fixed point; and, since the line representing Q is given in magnitude, AC is constant; hence, the angle CAO being variable, the locus of C, the extremity of the line which represents the force R, is a circle described about A as a centre with a radius AC.

(5) Three forces act on a point, their directions being parallel to the three perpendiculars drawn from the angles of a triangle to the opposite sides, and their magnitudes inversely

proportional to these perpendiculars: to prove that the three forces are in equilibrium.

Let ABC, fig. (13), be the triangle; AP, BQ, CR, the three perpendiculars; P, Q, R, the respective forces. Then, the two triangles BAQ, CAR, being similar,

$$BQ : CR :: AB : CA,$$

and therefore, the magnitudes of the forces being inversely as the perpendiculars,

$$Q : R :: CA : AB.$$

Similarly $$R : P :: AB : BC,$$

$$P : Q :: BC : CA,$$

or $$P : Q : R :: BC : CA : AB.$$

This proportion shews that the forces are in equilibrium.

(6) If three forces, the magnitudes of which are $3P$, $4P$, and $5P$, act at one point and are in equilibrium, shew that the forces $3P$ and $4P$ are at right angles to each other.

(7) Two weights are attached to given points of a fine string, the ends of which are tied to two fixed points: prove that the tensions of the three portions of the string cannot be all equal.

(8) Two small rings slide on the arc of a smooth vertical circle; a string passes through both rings and hangs below the arc, three equal weights being attached to it, one at each end and one on the portion between the rings; to find the position of the rings when they are in equilibrium.

If A, B, be the two rings, and C the position of the middle weight, ABC must be an equilateral triangle, of which the side AB is horizontal.

(9) A circular disc is kept at rest by three forces acting all outwards or all inwards along normals at three points of the circumference: shew that the forces are as the sides of the circumscribing triangle which touches the disc at those points.

SECT. 2. $P : Q : R :: \sin \widehat{Q, R} : \sin \widehat{R, P} : \sin \widehat{P, Q}$.

If three forces P, Q, R, act on a point, and $\widehat{Q, R}$, $\widehat{R, P}$, $\widehat{P, Q}$, denote the respective angles between Q, R; R, P; P, Q; the sufficient and necessary conditions of equilibrium are expressed by the following proportion

$$P : Q : R :: \sin \widehat{Q, R} : \sin \widehat{R, P} : \sin \widehat{P, Q}.$$

(1) From a fixed point O in a string AOB, fig. (14), which is fixed to the two points A, B, in a horizontal line, a weight W is suspended: to compare the tensions of the two strings AO, BO, α and β being their inclinations to the horizon.

Since $\angle OAB = \alpha$ and $\angle OBA = \beta$, it follows that

$$\angle AOW = \tfrac{1}{2}\pi + \alpha, \text{ and } \angle BOW = \tfrac{1}{2}\pi + \beta:$$

hence $$S : T :: \sin \angle BOW : \sin \angle AOW$$

$$:: \cos\beta : \cos\alpha.$$

COR. Since $\angle AOB = \pi - \alpha - \beta$, we see also that

$$S : W :: \cos\beta : \sin(\alpha + \beta),$$

and $$T : W :: \cos\alpha : \sin(\alpha + \beta).$$

(2) Three forces, the directions of which bisect the angles of a triangle, are in equilibrium; to shew that they are proportional to the cosines of the respective half-angles.

Let the directions of the three forces P, Q, R, intersect in the point O, fig. (15). Then, the forces being in equilibrium,

$$P : Q : R :: \sin \widehat{Q, R} : \sin \widehat{R, P} : \sin \widehat{P, Q}.$$

But $$\angle \widehat{Q, R} = \angle BOC = \pi - \frac{B + C}{2} = \frac{\pi}{2} + \frac{A}{2},$$

and therefore $$\sin \widehat{Q, R} = \cos \frac{A}{2}:$$

similarly, $\sin \widehat{R, P} = \cos \frac{B}{2}$,

and $\sin \widehat{P, Q} = \cos \frac{C}{2}$.

Hence $P : Q : R :: \cos \frac{A}{2} : \cos \frac{B}{2} : \cos \frac{C}{2}$.

(3) A hemispherical bowl ACB, fig. (16), contains a weight W which is attached to a weight P by means of a string passing over the rim of the bowl at a point A: to determine W's position of equilibrium.

Let M be the position of the weight's equilibrium, O being the centre of the sphere: let $\angle AOM = \theta$: then, the triangle AOM being isosceles, $\angle OMA = \frac{1}{2}(\pi - \theta)$.

The weight is kept at rest by the reaction R of the bowl, acting along MO, the force P, acting along MA, and the force W, acting vertically downwards: hence

$$\frac{P}{W} = \frac{\cos \theta}{\cos \frac{\theta}{2}},$$

$$2 \cos^2 \frac{\theta}{2} - 1 = \frac{P}{W} \cos \frac{\theta}{2},$$

$$\cos^2 \frac{\theta}{2} - \frac{P}{2W} \cos \frac{\theta}{2} = \frac{1}{2},$$

$$\cos \frac{\theta}{2} = \frac{P \pm (P^2 + 8W^2)^{\frac{1}{2}}}{4W}.$$

Since θ is evidently less than π and therefore $\frac{\theta}{2}$ than $\frac{\pi}{2}$, it follows that the negative sign in the numerator must be rejected: therefore the position of equilibrium is defined by the equation

$$\cos \frac{\theta}{2} = \frac{P + (P^2 + 8W^2)^{\frac{1}{2}}}{4W}.$$

If $P > W$, then $\cos \frac{\theta}{2} > 1$, and the weight could not rest in the bowl.

(4) Three forces P, Q, R, acting upon a particle, produce equilibrium: supposing the angle between P's direction and R's to be $\frac{2}{3}\pi$, and that between Q's and R's to be $\frac{1}{2}\pi$, to compare the magnitudes of the forces.

P, Q, R, are to each other respectively as 2, $\sqrt{3}$, 1.

(5) A sphere rests upon two inclined planes: to find the pressure which it exerts upon each.

Let W be the weight of the sphere; α, β, the inclinations of the two planes, and R, S, the respective pressures exerted upon them. Then

$$R = \frac{W \sin \beta}{\sin (\alpha + \beta)}, \quad S = \frac{W \sin \alpha}{\sin (\alpha + \beta)}.$$

(6) A boat B experiences a force F from the stream, parallel to the bank OA, and a force W from the wind, at right angles to OA towards the opposite bank: to find the tension of the rope OB by which the boat is tied to a point O in the bank, and to determine the inclination of OB to OA.

The tension of the rope is equal to

$$(F^2 + W^2)^{\frac{1}{2}},$$

and the angle AOB is equal to

$$\tan^{-1}\left(\frac{W}{F}\right).$$

(7) A weight is supported on an inclined plane, the inclination of which to the horizon is 30°, first, by a power parallel to the plane, and, secondly, by a power parallel to the base: to compare the pressures on the plane in the two cases.

The pressure in the former case is to that in the latter as 3 to 4.

(8) One end of a string AOP is attached to a fixed point A, and to the other end is fixed a weight P: a string BO, shorter than AB, is attached to a point B in the same horizontal line with A, and is connected with the string AOP by a moveable loop O, which is capable of sliding along AOP: to find the inclination of BO to the horizon, when there is equilibrium.

If $AB = a$, $BO = b$, and $\angle ABO = \phi$,

$$\cos\phi = \frac{b + (8a^2 + b^2)^{\frac{1}{2}}}{4a}.$$

(9) A string of length c is tied to two points A, B; the inclination of AB to the horizon is α; a weight, sliding freely on the string, hangs in equilibrium: shew that, if $AB = a$, the weight divides the string into two parts equal to

$$\tfrac{1}{2}c\left\{1 \pm \frac{a \sin\alpha}{(c^2 - a^2\cos^2\alpha)^{\frac{1}{2}}}\right\}.$$

(10) A string $AHKB$ is attached to two fixed points A, B, in a horizontal line: from fixed points H, K, in the string are suspended equal weights P, P: to find the tensions of the several portions of the string, supposing AH, HK, KB, to be all equal.

If $AB = c$, and $l =$ the whole length of the string, the tension of HK is equal to

$$\frac{P(3c - l)}{\{4l^2 - (3c - l)^2\}^{\frac{1}{2}}},$$

and that of AH or BK to

$$\frac{2Pl}{\{4l^2 - (3c - l)^2\}^{\frac{1}{2}}}.$$

(11) Two forces P and P', acting in the diagonals of a parallelogram, keep it at rest in such a position that one of its edges is horizontal: shew that

$$P \sec\alpha' = P' \sec\alpha = W \operatorname{cosec}(\alpha + \alpha'),$$

where W is the weight of the parallelogram, and α, α', are the angles at which its diagonals are inclined to the horizontal edge.

(12) A particle is placed in the centre of a circle and is acted on by three forces P, Q, R, tending towards the angular points A, B, C, of the circumscribed triangle: prove that, when there is equilibrium,

$$P : Q : R :: \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2},$$

where A, B, C, are the angles of the triangle.

(13) A smooth circular ring is fixed in a horizontal position, and a small ring P sliding upon it is in equilibrium when acted on by two strings in the direction of the chords PA, PB; shew that, if PC be a diameter of the circle, the tensions of the strings are in the ratio of BC to AC.

SECT. 3. *Each force equal and opposite to the resultant of the other two.*

If three forces act on a point, it is sufficient and necessary for equilibrium, that any one of them be equal and opposite to the resultant of the other two.

If P, Q, R, be the three forces, these conditions are equivalent to the equations

$$P^2 = Q^2 + R^2 + 2QR \cos \widehat{Q, R},$$

$$Q^2 = R^2 + P^2 + 2RP \cos \widehat{R, P},$$

$$R^2 = P^2 + Q^2 + 2PQ \cos \widehat{P, Q}.$$

(1) If a point O, fig. (17), be acted on by three forces represented in magnitude and direction by OA, OB, OC, O being the point of intersection of lines drawn from the angles of a triangle ABC to bisect opposite sides, to prove that the point will be in equilibrium.

Let P, Q, R, be the points of bisection of the sides: then the resultant of the forces OA, OB, is equal to a force $2OR$ along OR. But, by a known property of triangles, the line OC is double the line OR: hence the forces OA, OB, are equivalent to CO, and therefore balance OC.

(2) A force P, acting at an angle of 30° to an inclined plane, supports a weight W: supposing R to be the pressure on the plane, to prove that

$$W^2 = P^2 + PR + R^2.$$

The reaction of the inclined plane against the weight is equal to R, and the angle between this reaction and P's direction is equal to 60°: hence

$$\begin{aligned} W^2 &= P^2 + R^2 + 2PR \cos 60° \\ &= P^2 + PR + R^2. \end{aligned}$$

(3) Two weights P and Q hang at the ends of a string which passes over two smooth pegs A and B: a weight W is suspended from a point O at a point of the string between the pegs: to find W in terms of P and Q in order that the angle AOB may be a right angle.

The weight W is given by the relation

$$W^2 = P^2 + Q^2.$$

(4) Three forces act at the three angles of a triangle towards a point within the triangle and are proportional respectively to the distances of the point from the angles: give a geometrical construction for the position of the point in order that equilibrium may subsist.

CHAPTER III.

RESULTANT OF ANY NUMBER OF FORCES ACTING ON A POINT.

SECT. 1. *Geometrical method.*

(1) D is the middle point of the side BC of a triangle ABC, fig. (18): three forces, represented by AB, AC, DA, act upon the point A: to find the magnitude and direction of their resultant.

Complete the parallelogram $ABEC$ and join AE: the intersection of AE, BC, will coincide with the point D.

The resultant of AB, AC, is AE: hence the three forces AB, AC, DA, are equivalent to the two forces AE, DA, and therefore their resultant is the force DE or AD.

(2) ABC, fig. (19), being a triangle, the point A is acted upon by three forces, represented in magnitude and direction by the lines AB, AC, BC: to draw the line which shall represent their resultant in magnitude and direction.

Complete the parallelogram $BADC$. Then the two forces AB, BC, are equivalent to the two AB, AD, and therefore to the force AC: thus the three forces AB, AC, BC, are equivalent to double the force AC. Produce AC to E, making CE equal to AC. Then AE represents the resultant of the three forces AB, AC, BC, in magnitude and direction.

(3) A quadrilateral $ABCD$, fig. (20), is acted on by forces, which are represented in magnitude and direction by AB, AC, DB, DC, respectively: to prove that their resultant is represented in magnitude and direction by four times the distance between the middle points of AD, BC.

Let F, G, be the middle points of BC, AD, respectively. Join AF, DF, FG. Then the forces AB, AC, are equivalent

to $2AF$; and the forces DB, DC, to $2DF$. Again, the forces AF, DF, are equivalent to $2GF$. Hence the forces AB, AC, DB, DC, are equivalent to $4GF$.

(4) Three equal forces are represented in magnitude and direction by the lines drawn from the angles of a triangle to the centre of the circumscribing circle: to prove that the resultant is represented in magnitude and direction by the line joining that centre with the intersection of the three perpendiculars drawn from the angles of the triangle to the sides respectively opposite.

Let ABC, fig. (21), be the triangle, and O the centre of the circumscribed circle. Produce CO to meet the circle in F and join BF. Complete the parallelogram $BFOK$.

Then the force KO is equivalent to forces BO, KB, applied at O, that is, to forces BO, OF, that is, to the two forces BO, CO.

Let E be the intersection of the perpendiculars AP, BQ, CR, drawn from A, B, C, to BC, CA, AB, respectively.

Then BE, FA, are parallel, for $\angle FAC$, being an angle in a semicircle, is a right angle, and therefore equal to $\angle BQC$. Also BF, EA, are parallel, since $\angle FBC$, being an angle in a semicircle, is a right angle, and therefore equal to $\angle APC$. Hence $EA = BF = KO$: but EA, KO, are parallel: hence $OAEK$ is a parallelogram: hence the forces AO, KO, are equivalent to the force EO, that is, the forces AO, BO, CO, are equivalent to the force EO.

(5) AB, CD, are any two equal and parallel chords in a circle, and P is a point on the circumference, half way between A and B: prove that if forces, represented by the lines PA, PB, PC, PD, act at the point P, their resultant is constant.

(6) Forces are applied at one of the corners of a regular hexagon, acting towards the other corners, and are proportional in magnitude to the distances of those corners from the point of

application: to find the ratio which the magnitude of the resultant bears to that of one of the forces acting along a side.

The required ratio is that of 6 to 1.

(7) If O be the centre of the circle circumscribing a triangle ABC, and D, E, F, be the middle points of the sides; the system of forces represented by OA, OB, OC, will be equivalent to those represented by OD, OE, OF.

(8) A quadrilateral $ABCD$ is acted on by forces represented in magnitude and direction by AB, AD, CB, CD, respectively: prove that their resultant is represented in magnitude and direction by four times the line joining the middle points of the diagonals.

SECT. 2. *Trigonometrical method.*

(1) A force of two pounds and a force of three pounds act upon a point; the direction of the former force being inclined at an angle of 60°, and that of the latter at an angle of 45° to a given straight line passing through the point: to find the magnitude and direction of the resultant.

Let R = the magnitude of the resultant and θ = its inclination to the given straight line. Then

$$R \cos \theta = 2 \cos 60^\circ + 3 \cos 45^\circ = 1 + \frac{3}{\sqrt{2}},$$

and $$R \sin \theta = 2 \sin 60^\circ + 3 \sin 45^\circ = \sqrt{3} + \frac{3}{\sqrt{2}}.$$

Hence, $$R^2 = 13 + 3\sqrt{2}\,(1 + \sqrt{3}),$$

and $$\tan \theta = \frac{3 + \sqrt{6}}{3 + \sqrt{2}}.$$

(2) Three forces P, Q, R, acting on a point O, are inclined at angles α, β, γ, to a given line passing through O: to find the magnitude and direction of the resultant of these forces.

If θ = the inclination of the resultant to the given line,

$$\tan\theta = \frac{P\sin\alpha + Q\sin\beta + R\sin\gamma}{P\cos\alpha + Q\cos\beta + R\cos\gamma},$$

and the resultant is equal to the square root of

$$P^2 + Q^2 + R^2 + 2QR\cos(\beta-\gamma) + 2RP\cos(\gamma-\alpha) + 2PQ\cos(\alpha-\beta).$$

(3) Three forces, each equal to P, act on a point O, in directions OA, OB, OC; the angle AOC being a right angle, and the line OB bisecting the angle AOC: to find the magnitude of the resultant of the three forces.

The resultant force is equal to

$$P(1+\sqrt{2}).$$

(4) Three forces P, Q R, act on a point: R acts oppositely to P, and Q at right angles to both: to find the magnitude and direction of the resultant force.

The magnitude of the resultant force is equal to

$$\{(P-R)^2 + Q^2\}^{\frac{1}{2}},$$

and its angle of inclination to P's direction is equal to

$$\tan^{-1}\left(\frac{Q}{P-R}\right).$$

(5) A particle is placed in the middle point of a fixed horizontal, equilateral, and triangular board, and is kept in equilibrium by three equal weights, which act by means of strings passing through the angular points. Supposing the particle to be moved through a certain space towards one of the angular points, and there fixed; and that the resultant force exerted upon it by the strings is then equal to half of each of the weights, to find the inclinations of the strings to each other.

Let ABC be the triangle and suppose the particle to be displaced towards A: then, O being the new position of the particle, the angle AOB or AOC will be equal to

$$\cos^{-1}\left(-\frac{3}{4}\right).$$

CHAPTER IV.

EQUILIBRIUM OF ANY NUMBER OF FORCES ACTING ON A POINT.

SECT. 1. *Geometrical Method.*

(1) A circle, the plane of which is vertical, has a centre of constant repulsive force, equal to gravity, at one extremity of the horizontal diameter: to find the position of equilibrium of a particle within the circle.

Let P, fig. (22), be the position of equilibrium of the particle, A being the centre of force, and C the centre of the circle. Draw PQ vertically to intersect the horizontal diameter in Q.

Now, since the particle is acted on by two equal forces, the repulsive force along AP, and the force of gravity along QP, it is necessary to equilibrium that the reaction of the circle, which takes place along PC, bisect the angle between PA and PQ.

Hence $$\angle PCQ = \angle CAP + \angle APC$$
$$= 2 \angle APC = 2 \angle CPQ:$$

but $$2 \angle CPQ + 2 \angle PCQ = 180^\circ:$$

hence $$3 \angle PCQ = 180^\circ,$$

and therefore $$\angle PCQ = 60^\circ,$$

which determines the position of equilibrium.

(2) Shew that within a quadrilateral, no two sides of which are parallel, there is but one point at which forces, acting towards the corners and proportional to the distances of the point from them, can be in equilibrium.

Campion and Walton: *Solutions of the Cambridge Problems of* 1857.

SECT. 2. *Components along two lines at right angles to each other.*

(1) A weight W is sustained on an inclined plane by a certain force; the inclination of the force to the inclined plane is 30°, and the inclination of the plane to the horizon is 30°: to find the pressure on the inclined plane.

Let AB, fig. (23), be the inclined plane; let $R =$ the reaction of the plane against the weight, and $P=$ the sustaining force.

Then, resolving forces along and perpendicularly to the inclined plane, we have

$$P.\frac{\sqrt{3}}{2}=\frac{1}{2}W, \quad \text{or} \quad P=\frac{W}{\sqrt{3}} \text{..................(1)},$$

and
$$R+\frac{1}{2}P=W\frac{\sqrt{3}}{2} \text{.......................(2)}.$$

From (1) and (2), we see that

$$R=\frac{W}{\sqrt{3}}.$$

The figure (24) exhibits the two groups into which by resolution we have decomposed the three forces P, W, R; the equations (1) and (2) expressing the conditions for the equilibrium of the two new groups, each regarded as independent of the other.

(2) A, B, C, D, fig. (25), are four tacks forming the angles of a square, the lines AB and CD being horizontal: a fine string AEB hangs over the higher tacks, its lowest point E being at the centre of the square: another fine string $PDECP$, with equal weights P, P, at its ends, hangs over the former string and over the lower tacks: to find the tension of the string AEB and the pressures on all the tacks.

Let T be the tension of the string AEB. Then the point E is kept at rest by a force T along EA, a force T along EB, a force P along EC, and a force P along ED, that is, by one system of forces in the line AC, and another in the perpendicular line BD: hence $T=P$.

Again, the pressure on either of the higher tacks is equal to

$$2T\cos\frac{\pi}{8} = 2P\cos\frac{\pi}{8},$$

and, upon either of the lower, to

$$2P\cos\frac{3\pi}{8}.$$

(3) A string of given length is attached to the highest point O, fig. (26), of a given sphere: a weight P is fixed to the lower end of the string: to find the pressure on the hemisphere and the tension of the string.

Let OB be the arc touched by the string, and C the centre of the sphere: let $\angle BCO = \alpha$. Let R be the reaction of the sphere against the weight; this reaction will take place in the direction CB. Let T be the tension of the string: T will act on the weight tangentially to the sphere.

For the equilibrium of the weight we have, resolving forces tangentially and normally,

$$T = P\sin\alpha,$$

and $$R = P\cos\alpha.$$

Let c = the length of the string, and r = the radius of the sphere: then

$$T = P\sin\frac{c}{r},$$

and $$R = P\cos\frac{c}{r}.$$

(4) The ends of a fine cord of given length are tied to two given points in the same horizontal line, and a smooth ring, sliding on the cord, sustains a given weight: to find the tension of the cord.

Let A, B, fig. (27), be the two given points, ACB the cord in a position of rest: draw CH vertically to meet the horizontal line AB in H. Let W = the given weight, and T = the tension of the string. Let $\angle ACH = \theta$, $\angle BCH = \phi$. Then, since the

ring C is kept at rest by the tensions in the directions CA, CB, and the weight W, we have, resolving horizontally,

$$T \sin \theta = T \sin \phi \text{ .... (1)},$$

and, resolving vertically,

$$T \cos \theta + T \cos \phi = W \text{ .... (2)}.$$

From (1) we see that $\phi = \theta$, and therefore, by (2),

$$2T \cos \theta = W \text{ .... (3)}.$$

Let $2c =$ the length of the cord, and let $AB = 2a$: then $BC = c$, $BH = a$, and therefore

$$\sin \theta = \frac{a}{c}, \quad \cos \theta = \left(1 - \frac{a^2}{c^2}\right)^{\frac{1}{2}};$$

and, consequently, by (3),

$$T = \frac{Wc}{2(c^2 - a^2)^{\frac{1}{2}}}.$$

(5) A man, fig. (28), with a perfectly smooth spherical head, wears a conical hat: to find the whole pressure on his head.

Let $W =$ the weight and $2\alpha =$ the vertical angle of his hat, and let $R =$ the reaction of his head against his hat at any point of the circle of contact: this reaction will take place along the radius of his head at this point. Then $R \sin \alpha$ will denote the vertical component of R, and the sum of all such components must be equal to W: hence

$$\Sigma (R \sin \alpha) = W,$$

$$\Sigma (R) = \frac{W}{\sin \alpha}.$$

Hence the whole pressure on his head, which is equal to $\Sigma (R)$, is equal to

$$\frac{W}{\sin \alpha}.$$

COR. If α be exceedingly small, and W finite, the pressure

on his head will be enormous: thus a hat, although very light, would, if exceedingly tall, crack his head.

(6) A particle is placed on a square table at distances c_1, c_2, c_3, c_4, from the corners, and to it are attached strings passing over smooth pulleys at the corners and supporting weights P_1, P_2, P_3, P_4; to prove that, if there is equilibrium,

$$\frac{\dfrac{P_1}{c_1}-\dfrac{P_3}{c_3}}{c_1^2-c_3^2}=\frac{\dfrac{P_2}{c_2}-\dfrac{P_4}{c_4}}{c_2^2-c_4^2}.$$

Let O, fig. (29), be the position of the particle; C the centre of the square $A_1A_2A_3A_4$: draw CM, OM, parallel to A_4A_1, A_1A_2, to intersect in M: let $CM=x$, $OM=y$. Let

$$OA_1=c_1,\quad OA_2=c_2,\quad OA_3=c_3,\quad OA_4=c_4,$$

and let $2a$ represent the length of a side of the square.

For the equilibrium of the particle, we have, resolving forces parallel and perpendicularly to CM,

$$P_1.\frac{a-x}{c_1}+P_2.\frac{a-x}{c_2}=P_3.\frac{a+x}{c_3}+P_4.\frac{a+x}{c_4},$$

and $$P_1.\frac{a-y}{c_1}+P_4.\frac{a-y}{c_4}=P_2.\frac{a+y}{c_2}+P_3.\frac{a+y}{c_3}.$$

These equations may be written in the form

$$x\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}+\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)=a\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}-\frac{P_3}{c_3}-\frac{P_4}{c_4}\right),$$

$$y\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}+\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)=a\left(\frac{P_1}{c_1}-\frac{P_2}{c_2}-\frac{P_3}{c_3}+\frac{P_4}{c_4}\right),$$

whence, by addition,

$$(x+y)\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}+\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)=2a\left(\frac{P_1}{c_1}-\frac{P_3}{c_3}\right)\;\dots\dots\dots\;(1).$$

Now from the geometry it is plain that

$$c_1^2=(a-x)^2+(a-y)^2,$$

$$c_3^2=(a+x)^2+(a+y)^2;$$

hence $$c_1^2 - c_3^2 = -4a(x+y):$$

and the equation (1) becomes

$$-\frac{1}{8a^2}\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}+\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)=\frac{\frac{P_1}{c_1}-\frac{P_3}{c_3}}{c_1^2-c_3^2}.$$

By symmetry it is evident that we must also have

$$-\frac{1}{8a^2}\left(\frac{P_1}{c_1}+\frac{P_2}{c_2}+\frac{P_3}{c_3}+\frac{P_4}{c_4}\right)=\frac{\frac{P_2}{c_2}-\frac{P_4}{c_4}}{c_2^2-c_4^2}.$$

Hence, finally,

$$\frac{\frac{P_1}{c_1}-\frac{P_3}{c_3}}{c_1^2-c_3^2}=\frac{\frac{P_2}{c_2}-\frac{P_4}{c_4}}{c_2^2-c_4^2}.$$

(7) A weight W is supported on an inclined plane AB, fig. (30), by two forces, each equal to $\frac{W}{n}$, one of which acts parallel to the base AC: to determine the conditions of equilibrium.

Let α be the inclination of the plane, R the reaction, and ϵ the inclination of the other supporting force to the plane.

Resolving forces parallel to AB, we have

$$\frac{W}{n}\cos\epsilon+\frac{W}{n}\cos\alpha=W\sin\alpha,$$

$$\text{or } \cos\epsilon+\cos\alpha=n\sin\alpha\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Resolving forces at right angles to AB, we have

$$R+\frac{W}{n}\sin\epsilon=W\cos\alpha+\frac{W}{n}\sin\alpha,$$

$$\text{or } R=W\left(\cos\alpha+\frac{\sin\alpha}{n}-\frac{\sin\epsilon}{n}\right)\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) we see that

$$1 - 2\sin^2\frac{\epsilon}{2} + 2\cos^2\frac{\alpha}{2} - 1 = 2n\sin\frac{\alpha}{2}\cos\frac{\alpha}{2},$$

$$\sin^2\frac{\epsilon}{2} = \cos^2\frac{\alpha}{2} - n\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$= n\cos^2\frac{\alpha}{2}\left(\frac{1}{n} - \tan\frac{\alpha}{2}\right) \qquad\qquad(3).$$

This result shews that the equilibrium is impossible if α be greater than $2\tan^{-1}\frac{1}{n}$. If α be less than $2\tan^{-1}\frac{1}{n}$, the equation (3) gives us two equal values of ϵ with opposite signs: thus the force, of which the direction is not given, may act as indicated in the diagram or at an equal inclination below the inclined plane.

Again, the equations (1) and (2) may be written in the forms

$$\cos\epsilon = n\sin\alpha - \cos\alpha,$$

$$\sin\epsilon = n\cos\alpha + \sin\alpha - \frac{nR}{W}:$$

squaring these two equations and then adding, we get

$$1 = n^2 + 1 + \frac{n^2R^2}{W^2} - \frac{2nR}{W}(n\cos\alpha + \sin\alpha),$$

whence $\qquad R^2 - \frac{2}{n}W.R.(n\cos\alpha + \sin\alpha) = -W^2,$

and therefore

$$R = W\left(\cos\alpha + \frac{\sin\alpha}{n}\right) \pm W\left\{\left(\cos\alpha + \frac{\sin\alpha}{n}\right)^2 - 1\right\}^{\frac{1}{2}},$$

which determines the magnitude of the reaction of the inclined plane against the weight W.

The values of R here given will be both possible and positive if $\cos\alpha + \frac{\sin\alpha}{n}$ be greater than 1, that is, if

$$\sin\alpha > n(1 - \cos\alpha),$$

$$\text{or } \tan\frac{\alpha}{2} < \frac{1}{n}, \quad \text{or } \alpha < 2\tan^{-1}\frac{1}{n}.$$

The sign + or − must be adopted in the expression for R accordingly as the sign − or + is assigned to ϵ.

(8) Prove that the forces $3P\sqrt{2}$, $4P$, $5P$, $7P$, $2P$, acting in one plane upon a point, the directions of the last four making angles of 45°, 135°, 225°, 315°, respectively with that of the first, will produce equilibrium.

(9) A weight of 10 pounds is placed on an inclined plane, the angle of which is 30°: to find the magnitude of a horizontal force which will sustain the weight.

The required force is equal to $\frac{10}{\sqrt{3}}$ pounds.

(10) Two forces P, Q, acting respectively parallel to the base and length of an inclined plane, will each of them singly sustain upon it a particle of weight W: prove that

$$W = \frac{P \cdot Q}{(P^2 - Q^2)^{\frac{1}{2}}}.$$

(11) A weight W is supported on an inclined plane AB, fig. (31), by three forces, each equal to P, one acting vertically upwards, another parallel to the horizontal line AC, and the third along AB: to find the inclination of the plane.

The required inclination is equal to

$$2\tan^{-1}\left(\frac{P}{W-P}\right).$$

(12) If P be the force which, acting along an inclined plane, will support a weight W, R being the pressure on the plane; prove that, P' being the force which, acting horizontally, will support the same weight,

$$P : R :: P' : W.$$

(13) A power P supports a weight W on a plane, the inclination of which is α; prove that, if the pressure on the plane be

equal to P, the angle which the direction of the power makes with the plane is equal to

$$\frac{1}{2}(\pi - 4\alpha).$$

(14) Two heavy particles, P and Q, are connected together by a fine thread passing over a smooth pully at C: P rests on a smooth inclined plane AB, and Q hangs freely: prove that, α denoting the inclination of the plane to the horizon, R the pressure, and θ the angle CPB,

$$\cos\theta = \frac{P\sin\alpha}{Q}, \quad R = P\cos\alpha - (Q^2 - P^2\sin^2\alpha)^{\frac{1}{2}}.$$

(15) A weight W is supported on an inclined plane by a certain force acting at an angle ϵ to the plane, the inclination of the plane being α: the same weight is sustained on the same plane by the same force acting at an angle $\frac{1}{2}\pi - \epsilon$: prove that in each case the pressure on the plane is equal to

$$W(\cos\alpha - \sin\alpha).$$

(16) A weight of 6 lbs. is placed on an inclined plane, the height of which is 3 feet and base 4 feet, and is attached by a string to an equal weight hanging over the top of the plane: find how much must be added to the weight on the plane that there may be equilibrium, and determine the pressure on the plane.

The amount which must be added is 4 lbs., and the pressure on the plane is 8 lbs.

(17) If two planes have the same height, and if two weights balance on them by means of a string which passes over the common vertex, prove that the weights will be to one another as the lengths of the planes.

(18) Two weights of 3 pounds and 5 pounds rest on a double-inclined plane, being connected together by a fine string which passes over the common summit of the two planes: supposing the inclination of the plane, on which the former weight rests, to be 30°, find the inclination of the other plane.

The required inclination is equal to

$$\sin^{-1}\left(\frac{3}{10}\right).$$

(19) Two equal weights P, P, attached to the ends of a string, which passes over two pegs A, B, fig. (32), in a horizontal line, sustain a weight W suspended at the end of a string the higher end O of which is fixed to a point in the former string: find the magnitude of the angle AOB, when there is equilibrium, and ascertain whether it is possible for O to lie in the line AB.

If $\angle AOB = 2\theta$, then $\cos\theta = \frac{W}{2P}$, a result which shews that, unless P be infinite, O must lie below the line AB.

(20) A weight W, fig. (33), is sustained by a string DW, the upper end of which is looped to a string $ABDCA$, which hangs over three tacks A, B, C, fixed at the angles of an equilateral triangle, B and C being in a horizontal line: find the pressure on the tacks, the portion BDC of the string being supposed to be equal to the portion BAC.

The pressure on A is equal to W, and that on B or C to $\frac{W}{\sqrt{3}}$.

(21) A string, of length c, passes over four tacks forming a square $ABCD$, fig. (34), of which two sides are horizontal, the length of each side being a: a weight W is suspended by a string hanging from the former one by a loop at E: find the tension of the former string.

The required tension is equal to

$$\frac{\frac{1}{2}W(c-3a)}{(c^2-6ac+8a^2)^{\frac{1}{2}}}.$$

(22) A string, the extremities of which are fastened to the ends of a uniform bar of known weight, passes over four tacks so as to form with the bar a regular hexagon, the bar being hori-

zontal: find the vertical component of the pressure upon either of the two highest tacks.

The required vertical component is equal to half the weight of the bar.

SECT. 3. *Components along any two lines not parallel.*

(1) To prove that three equal forces, represented by OA, OB, OC, fig. (35), which act on a point O, may be balanced by a single force LO, L being the point of intersection of the perpendiculars from the angles of the triangle ABC on the opposite sides.

The sum of the components of the three equal forces OA, OB, OC, parallel to BC, is represented by the projection of OA upon BC: but this projection is equal to that of OL, and therefore the sum of the components of OA, OB, OC, parallel to BC is equal and opposite to the component of LO parallel to this same line.

The analogous proposition is true in regard to either CA or AB.

Hence the four forces OA, OB, OC, LO, will be in equilibrium.

COR. The resultant of the three equal forces OA, OB, OC, is represented in magnitude and direction by OL.

SECT. 4. *Friction.*

(1) If the roughness of a plane, which is inclined to the horizon at a known angle, be such that a body will just rest on it, to find the least force along the plane requisite to drag the body up.

Let α be the inclination of the plane, W the weight of the body, R the reaction of the plane, P the required force, μ the

coefficient of friction. Then, resolving forces along and perpendicularly to the plane, we have

$$P = \mu R + W \sin \alpha,$$

and $$R = W \cos \alpha,$$

and therefore $$P = (\mu \cos \alpha + \sin \alpha)\ W.$$

But, since the body just rests on the plane without support, $\mu = \tan \alpha$: hence

$$P = 2W \sin \alpha.$$

(2) A body is supported on a rough inclined plane by a force acting along the plane: supposing the greatest magnitude of the force to be double the least magnitude, to determine the inclination of the plane to the horizon.

Let W be the weight of the body, R the reaction of the plane; P the least magnitude of the force, and accordingly $2P$ the greatest. Let μ represent the coefficient of friction and α the inclination of the plane. The forces producing equilibrium in the two cases are exhibited in the figures (36) and (37).

For equilibrium, in the first case, we have, resolving forces parallel and perpendicularly to the plane,

$$P + \mu R = W \sin \alpha,$$

and $$R = W \cos \alpha;$$

whence $$P + \mu W \cos \alpha = W \sin \alpha \text{....................(1).}$$

The corresponding equation in relation to the second figure, replacing P by $2P$ and μ by $-\mu$, is

$$2P - \mu W \cos \alpha = W \sin \alpha \text{....................(2).}$$

Multiplying (1) by 2, and subtracting (2) from the resulting equation, we have

$$3\mu W \cos \alpha = W \sin \alpha,$$

and therefore $$\tan \alpha = 3\mu,$$

or $$\alpha = \tan^{-1}(3\mu).$$

(3) A heavy body is kept at rest on a given inclined plane by a force making a given angle with the plane; to prove that the

reaction of the plane, when it is smooth, is an harmonic mean between the greatest and least normal reactions, when it is rough.

Let W be the weight of the body, P the supporting force, R the normal reaction, λR the frictional reaction estimated up the plane; also let α be the inclination of the plane, and ϵ the inclination of P to the plane.

Then, resolving along and at right angles to the plane, we have

$$P \cos \epsilon + \lambda R = W \sin \alpha,$$

and
$$P \sin \epsilon + R = W \cos \alpha.$$

Eliminating P between these equations, we have

$$R (\cos \epsilon - \lambda \sin \epsilon) = W \cos (\alpha + \epsilon),$$

$$\frac{1}{R} = \frac{\cos \epsilon - \lambda \sin \epsilon}{W \cos (\alpha + \epsilon)}.$$

Hence, μ being the coefficient of friction, and R', R'', the values of R when λ is equal $+\mu$, $-\mu$, respectively, which are the extreme limits of the value of λ,

$$\frac{1}{R'} = \frac{\cos \epsilon - \mu \sin \epsilon}{W \cos (\alpha + \epsilon)}, \quad \frac{1}{R''} = \frac{\cos \epsilon + \mu \sin \epsilon}{W \cos (\alpha + \epsilon)},$$

and therefore

$$\frac{1}{R'} + \frac{1}{R''} = \frac{2 \cos \epsilon}{W \cos (\alpha + \epsilon)}$$

$$= \frac{2}{R_{,}},$$

$R_{,}$ being the reaction when the plane is smooth.

(4) P is the lowest point on the rough circumference of a circle, in a vertical plane, at which a particle P can rest, the coefficient of friction being unity: determine the inclination of the radius through P to the horizon.

The required inclination $= \dfrac{\pi}{4}$.

(5) A given horizontal force P just supports a weight W on a rough plane inclined to the horizon at an angle θ: the same

weight will just rest without support on a plane of the same material, when the inclination is α: determine θ.

The angle θ is given by the equation

$$\tan\theta = \frac{P + W\tan\alpha}{W - P\tan\alpha}.$$

(6) A weight is to be conveyed to the top of a rough plane, inclined to the horizon at an angle α: prove that, if the coefficient of friction be greater than

$$\sec\alpha - \tan\alpha,$$

it will be easier to lift the weight than to drag it up by means of a cord parallel to the plane.

(7) If the ratio of the greatest to the least force, acting parallel to a given rough inclined plane, which will support a certain weight on the plane, be equal to the ratio of the weight to the normal pressure on the plane, determine the coefficient of friction.

If α be the inclination of the plane to the horizon, the coefficient of friction is equal to

$$\tan\alpha \, . \tan^2\frac{\alpha}{2}.$$

(8) A body is just supported on a rough inclined plane by a force acting along it; find the angle between this plane and a smooth plane, on which the body may be supported by the same force acting at the same angle to the horizon as before.

If μ denote the coefficient of friction, the required angle is equal to $\tan^{-1}(\mu)$.

(9) A particle, the weight of which is W, rests upon a rough horizontal table, and is acted upon by two horizontal forces mW and nW, the directions of which are at right angles to each other; find the least value of the coefficient of friction consistent with the equilibrium of the particle.

The least value of the coefficient of friction is equal to

$$(m^2 + n^2)^{\frac{1}{2}}.$$

(10) If P just supports W on a rough inclined plane, and the plane is depressed through an angle the tangent of which is the coefficient of friction; prove that, if friction suddenly ceases, there will still be equilibrium, P's direction remaining unaltered.

(11) A rough right-angled triangle stands on its hypotenuse, and equal particles, connected by a fine string passing round a small smooth pully at the vertex, so rest on the sides that one is but just supported: after the string is cut, the other is but just supported; find the angles at the hypotenuse.

The angles at the hypotenuse are $\frac{3}{8}\pi$ and $\frac{1}{8}\pi$.

(12) Two weights W, W', of different materials, connected by a fine string which passes over a smooth peg, are placed on a rough inclined plane passing through the peg, the inclination of the plane being α: the two parts of the string are respectively perpendicular and inclined at an angle β to the intersection of the inclined plane with a horizontal plane: the weights are in equilibrium, the former being only just supported: prove that, μ, μ', being the respective coefficients of friction between the weights and the inclined plane,

$$W^2(\tan\alpha-\mu)^2-2WW'\tan\alpha\sin\beta(\tan\alpha-\mu)$$
$$+W'^2(\tan^2\alpha-\mu'^2)=0.$$

CHAPTER V.

EQUILIBRIUM OF A BODY MOVEABLE ABOUT A FIXED AXIS.

If two forces act upon an imponderable lever, and produce equilibrium, their resultant must pass through the fulcrum, and, conversely, if their resultant passes through the fulcrum, there will be equilibrium.

The above condition of equilibrium may be replaced by the following one: viz. that the moments of the forces about the fulcrum shall be equal and opposite.

If more than two forces act on the lever and produce equilibrium, the resultant of all the forces must pass through the fulcrum, and, conversely, if their resultant passes through the fulcrum, there will be equilibrium.

The above condition may be replaced by the following one: viz. that the sum of the moments of all the forces which tend to twist the lever in one direction about the fulcrum shall be equal to the sum of the moments of all the forces which tend to twist the lever about the fulcrum in the opposite direction.

SECT. 1. *Body acted on by two forces.*

(1). If AB, fig. (38), be a rigid imponderable rod, moveable about A, and be in equilibrium under the action of any two forces represented by BF, BC; to shew that, if FB be produced to a point D such that $BD = BF$, and C, D, be joined, CD is parallel to BA.

Join CD: then the resultant of the two forces BF, BC, that is, of DB, BC, will be a force, equal and parallel to DC, acting on the point B. Now a single force acting on B cannot keep AB

at rest about the fulcrum A unless its direction coincides with AB: the line DC must therefore be parallel to AB.

(2) A shop-keeper has in his shop correct weights but a false balance, that is, one having one arm longer than the other: supposing that he serves out to each of two customers articles weighing, as indicated by his balance, W pounds, using for the commodities first one scale and then the other; to find whether he gains or loses by the peculiarity of his balance.

Let a, b, be the lengths of the arms of the balance, and let P, Q, be the actual weights of the two commodities, placed at the ends of the former and latter arm respectively.

Then $$Pa = Wb,$$

and $$Qb = Wa.$$

Hence, $$P + Q - 2W = W\left(\frac{b}{a} + \frac{a}{b} - 2\right)$$
$$= W \cdot \frac{(a-b)^2}{ab}.$$

The shop-keeper therefore loses by his sale to the two customers a weight of substance equal to

$$W \cdot \frac{(a-b)^2}{ab}.$$

(3) A uniform bent rod ABC, fig. (39), moveable about its lower end A, which is fixed, is in equilibrium: to find the inclination of AB to the horizon, ABC being a right angle.

Let $AB = 2a$, $BC = 2b$; then the weights of AB, BC, may be represented by λa, λb, respectively. Let $\angle BAX = \theta$, AX being a horizontal line.

Conceive a beetle to crawl from A to the middle point of AB: it will have arrived at the centre of gravity of AB, having travelled horizontally leftwards through a space $a \cos\theta$; thus the moment of AB about A will be $\lambda a \cdot a \cos\theta$. Again, conceive a beetle to crawl along the rod from A to the middle point of BC, that is, to the centre of gravity of BC: in going from A to B it will have travelled horizontally leftwards through a space

$2a \cos \theta$, and, in going from B to the middle point of BC, it will have travelled horizontally rightwards through a space $b \sin \theta$: thus altogether the beetle will have travelled horizontally leftwards through a space $2a \cos \theta - b \sin \theta$, and accordingly the moment of BC's weight about A is equal to

$$\lambda b\,(2a \cos \theta - b \sin \theta).$$

Thus, for the equilibrium of the rod we have, taking moments about A,

$$\lambda a \,.\, a \cos \theta + \lambda b\,(2a \cos \theta - b \sin \theta) = 0,$$

and therefore $$\tan \theta = \frac{a^2 + 2ab}{b^2}.$$

(4) Two forces P and Q act at the ends A and B, respectively, of a weightless straight lever AB: to find the position of the fulcrum that equilibrium may be maintained, the inclinations of P's and Q's directions to AB being α and β respectively.

Let $AB = c$; and let x, y, be the distances of the fulcrum from A, B, respectively. Then, taking moments about the fulcrum, we get

$$P \,.\, x \,.\, \sin \alpha = Q \,.\, y \,.\, \sin \beta:$$

but $$x + y = c:$$

hence $$x\,(P \sin \alpha + Q \sin \beta) = Qc \sin \beta,$$

and therefore $$x = \frac{Qc \sin \beta}{P \sin \alpha + Q \sin \beta};$$

and similarly $$y = \frac{Pc \sin \alpha}{P \sin \alpha + Q \sin \beta}.$$

(5) A rod AB, fig. (40), moveable about a smooth hinge at A, is attached, by means of a fine string passing over a fixed pully C, to a weight P: to find the position of the rod when in equilibrium.

Let the horizontal line through A be intersected by CP in D and by CB, produced, in E. Let $W =$ the weight of AB, $AB = 2a$, $AD = h$, $CD = k$, $\angle BAE = \theta$, $\angle AEB = \phi$.

Taking moments about A, for the equilibrium of AB, we have

$$P \,.\, 2a \sin ABC = Wa \cos BAE,$$

and therefore $\quad 2P \sin(\theta+\phi) = W\cos\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$

Again, from the geometry it is plain that D's distance from BC is equal to either $k \cos\phi$, or to $h \sin\phi + 2a \sin(\theta+\phi)$: hence

$$k \cos\phi = h \sin\phi + 2a \sin(\theta+\phi) \quad\ldots\ldots\ldots (2).$$

The equations (1) and (2) determine the magnitudes of the angles θ and ϕ, and thus define the position of AB.

(6) A uniform rod CD, fig. (41), moveable about a smooth hinge C, presses against a given inclined plane AB: to find the inclination of CD to the horizon in order that the pressure exerted by it on the plane may be equal to half its own weight.

Let R be the reaction of the plane against the rod, and W the weight of the rod. Let $\alpha =$ the inclination of the plane and θ of the rod to the horizon.

Then, for the equilibrium of the rod, taking moments about C, we have

$$R \,.\, CD \cos(\alpha-\theta) = W \,.\, CG \cos\theta:$$

but, the rod being uniform, $CG = \frac{1}{2} CD$, and, by the hypothesis,

$$R = \tfrac{1}{2} W:$$

hence $\quad \cos(\alpha-\theta) = \cos\theta,$

$$\alpha - \theta = \theta,$$

$$\theta = \tfrac{1}{2}\alpha.$$

(7) When a boy of weight B ascends to a point b of a ladder AB, fig. (42), the lower end of which is fixed while the higher rests against a smooth vertical wall CD, its pressure against the wall is the same as when a man of weight M ascends to a point m: to prove that

$$B : M :: Am : Ab.$$

Let G be the centre of gravity of the ladder and W its weight: let R denote the reaction of the wall against the ladder, in both cases. Let $\angle BAC = \alpha$.

Then, when the boy is on the ladder,

$$R . BC = W . AG . \cos\alpha + B . Ab . \cos\alpha:$$

and, when the man is on the ladder,

$$R . BC = W . AG . \cos\alpha + M . Am . \cos\alpha:$$

from these two equations it is evident that

$$B . Ab = M . Am,$$

or that $$B : M :: Am : Ab.$$

(8) A lever without weight in the form of the arc of a circle, having two weights P and Q suspended from its extremities, rests with its convexity downwards upon a horizontal plane; to determine the position of equilibrium.

Let AEB, fig. (43), be the arc, resting at the point E on a horizontal plane. Let C be the centre of the whole circle: join CA, CB, and draw the radius CD to bisect the angle ACB. Join CE. Let $\angle ACD = \alpha = \angle BCD$, and $\angle ECD = \theta$.

Taking moments about E, as a fulcrum, we have, for the equilibrium of the lever,

$$P . AC \sin(\alpha - \theta) = Q . BC . \sin(\alpha + \theta),$$

and therefore

$$P(\tan\alpha - \tan\theta) = Q\{\tan\alpha + \tan\theta),$$

$$\tan\theta = \frac{P - Q}{P + Q} . \tan\alpha.$$

This equation determines the value of θ, and therefore defines the position of equilibrium.

(9) A uniform rod, of weight W and length 9 inches, rests on a fulcrum placed at the distance of 3 inches from one of its ends: find what weight must be suspended from that end to balance the longer arm.

The required weight is equal to $\frac{1}{2} W$.

(10) A power P, acting vertically downwards at one end of a straight lever, three feet from the fulcrum, balances a weight $3P$ placed at the other end: find the length of the lever, neglecting its weight.

The length of the lever is 4 feet.

(11) ABC is a straight rod, B being its middle point: if AB be of uniform density, and BC be without weight, what weight must be hung at C to keep the rod at rest about B as a fulcrum?

The required weight must be equal to half that of the rod.

(12) It is observed that a beam AB, the length of which is 12 feet, will balance at a point 2 feet from the end A; but, when a weight of 100lbs. is hung from the end B, it balances at a point 2 feet from that end: find the weight of the beam.

The required weight is 25lbs.

(13) ABC is a straight weightless lever, 9 inches long, placed between two pegs at A and B, 4 inches apart, so as to be kept horizontal by means of them and a weight of 10lbs. hanging at C: find the pressures on the pegs.

The pressure on the pegs A, B, are respectively 12lbs. 8oz., and 22lbs. 8oz.

(14) The arms of a balance are unequal, and a substance, placed successively in each scale, appears to weigh P and Q pounds; prove that the lengths of the arms are as $P^{\frac{1}{2}} : Q^{\frac{1}{2}}$.

(15) If the weights P and Q, P being the greater, balance on a horizontal straight lever ACB about a fulcrum at C, and the weights be interchanged so that Q acts at A and P at B; shew that the additional weight required at A to maintain equilibrium will be equal to

$$\frac{P^2 - Q^2}{Q}.$$

(16) It is found that a body weighs P, when suspended at the end A of a false balance AB without weight, and Q, when suspended at B: prove that the fulcrum ought to be shifted towards A through a space equal to

$$\frac{P^{\frac{1}{2}} - Q^{\frac{1}{2}}}{P^{\frac{1}{2}} + Q^{\frac{1}{2}}} \cdot \frac{AB}{2}.$$

(17) The lever of a false balance is 3 feet long, and, if a certain body is placed in one scale, it weighs 4 lbs., and, in the other, weighs 6 lbs. 4 oz.: find the true weight of the body and the lengths of the arms of the balance.

The true weight of the body is 5 lbs., and the lengths of the arms are 1 ft. 4 in. and 1 ft. 8 in.

(18) The arms of a balance are unequal, and one of the scales is loaded; a body, the true weight of which is P pounds, appears, when placed in the loaded scale, to weigh W pounds, and, when placed in the other scale, to weigh W' pounds: find the ratio of the arms and the weight with which the scale is loaded.

The ratio of the arms is equal to $\dfrac{W-P}{P-W'}$, and the weight required is equal to

$$\frac{P^2-WW'}{W-P}.$$

(19) A false balance has one of its arms 10 inches and the other $9\frac{1}{2}$ inches in length: by always putting the weight into the scale with the shorter arm, how much does the shop-keeper gain in every cwt.?

He gains one twentieth of a hundred-weight.

(20) An isosceles triangle ACB rests, with its axis vertical, on its vertex C: weights P, Q, are suspended from A, D, respectively, D being the middle point of BC: find the ratio of P to Q.

$$P : Q :: 1 : 2.$$

(21) A uniform lever, with its arms AC and BC at right angles, is in a position of equilibrium: find the angle which AC makes with the horizon,

If $AC = a$, $BC = b$, the required angle is equal to

$$\tan^{-1}\left(\frac{a^2}{b^2}\right).$$

(22) ABC is a uniform bent rod, the two parts being perpendicular to each other: supposing the rod to rest upon A as

a fulcrum, when the inclination of AB to the horizon is $\tan^{-1} 8$, prove that

$$AB = 2BC.$$

(23) A triangular lamina ABC, the weight of which is W, is suspended at the point C: find the weight which must be attached to B, that the vertical through C shall bisect the angle ACB.

If $BC = a$, and $AC = b$, the required weight is equal to

$$\frac{W}{3} \cdot \frac{b-a}{a}.$$

(24) A weightless lever, bent at right angles, with the angle for fulcrum, and having one arm double the other, has two equal weights hanging from its ends: if the perpendicular distance of the fulcrum from the direction of either weight be one inch, find the lengths of the arms.

The lengths of the arms in inches are $\frac{1}{2}\sqrt{5}$ and $\sqrt{5}$.

(25) One end of a uniform beam is placed on the ground against a fixed obstacle, and to the other is attached a string, which runs in a horizontal direction to a fixed point vertically above the obstacle, and, passing freely over it, sustains a weight W at its extremity, the beam being thus held at rest at an inclination of 45° to the horizon: prove that, if the string were attached to the centre instead of the end of the beam, and passed over the same fixed point, a weight, at the end of the string, equal to $W\sqrt{2}$, would keep the beam at rest in the same position.

(26) If m lbs. at one extremity of a log balance it upon a prop, n lbs. at the same extremity, when the prop is removed p feet, and r lbs., when it is removed q feet, further from that end; find the weight of the log.

The required weight is equal to

$$\frac{\dfrac{np}{m-n} - \dfrac{rq}{m-r}}{\dfrac{q}{m-r} - \dfrac{p}{m-n}}.$$

SECT. 2. *Body acted on by any number of forces.*

(1) AB, fig. (44), is a straight rod moveable about its lower end A: CD and EF are two other rods of the same thickness as AB, fixed to AB at right angles: to find at what point of AB a given horizontal force must act, so as to keep AB at rest in a vertical position.

Let $CD = 2b$, $EF = 2c$, $x =$ the distance of the required point from A. Let P, Q, denote the weights of CD, EF, respectively, and let R denote the given force.

Then, taking moments about A, we see that

$$R \, . \, x + Q \, . \, \tfrac{1}{2} EF = P \, . \, \tfrac{1}{2} \, CD.$$

Let $2h$ represent the length of a rod, of the same thickness as AB, the weight of which shall be equal to R: then

$$hx + c^2 = b^2,$$

$$x = \frac{b^2 - c^2}{h}.$$

(2) To investigate the inclination of AB to the horizon in the preceding question in order that the system of rods may remain at rest without the aid of any subsidiary force.

Let θ be the inclination of AB, fig. (45), to the horizon.

Let $AB = 2a$, $AC = k$, $AE = l$, $W =$ the weight of AB, the rest of the notation being the same as in the preceding question. Then, taking moments about A, we have

$$W \, . \, a \cos\theta + P(k\cos\theta - b\sin\theta) + Q(l\cos\theta + c\sin\theta) = 0,$$

and therefore

$$\tan\theta = \frac{Wa + Pk + Ql}{Pb - Qc}$$

$$= \frac{a^2 + bk + cl}{b^2 - c^2}.$$

(3) Supposing the system of rods, described in the two preceding questions, to rest with the end D of CD in contact

with a fixed horizontal plane through A, fig. (46); to find the pressure on this plane.

Let S denote the mutual action of the horizontal plane and the rod CD; and let $\angle BAD = \alpha$; then, for the equilibrium of the system of rods, we have, taking moments about A and retaining the notation of the two preceding questions,

$$S.AD = W.\tfrac{1}{2}AB.\cos\alpha + P.(AC.\cos\alpha + \tfrac{1}{2}CD.\sin\alpha)$$
$$+ Q(AE.\cos\alpha - \tfrac{1}{2}EF.\sin\alpha),$$

and therefore, multiplying both sides by AD,

$$S.(4b^2 + k^2) = Wak + P(k^2 + 2b^2) + Q(kl - 2bc),$$

or, if $2y$ denote the length of a rod, the thickness of which is the same as that of any one of the rods and the weight equal to S,

$$y = \frac{a^2k + b(2b^2 + k^2) + c(kl - 2bc)}{4b^2 + k^2}.$$

(4) A hornet, a toad, a mouse, and a nightingale, rest at the four angles of a rectangular board, the plane of which is vertical, and which is supported at its central point: to find its position of equilibrium.

Let the weights of these creatures be represented by H, T, M, N, respectively, and let them rest at the angles H, T, M, N, fig. (47), of the rectangle. Let O be the intersection of the diagonals HM, TN, of the rectangle. Let Ox be an indefinite horizontal line through O.

$$\text{Let } \angle HON = \alpha, \quad \angle HOx = \theta.$$

Taking moments about O for the equilibrium of the system, we get

$$H.OH\cos\theta + N.ON\cos(\alpha - \theta)$$
$$= M.OM\cos\theta + T.OT\cos(\alpha - \theta),$$

and therefore

$$(N - T).\cos(\alpha - \theta) = (M - H).\cos\theta,$$

$$\cos\alpha + \sin\alpha . \tan\theta = \frac{M-H}{N-T},$$

$$\tan\theta = \frac{M-H}{N-T} . \operatorname{cosec}\alpha - \cot\alpha.$$

(5) A uniform rod of length l is cut into three portions a, b, c, and these are formed into a triangle; when the triangle is placed in unstable equilibrium, resting with its plane vertical, one of its angular points being supported upon a smooth horizontal plane, to find the angle which the uppermost side makes with the horizon; and to shew that, if α, β, γ, be the three angles corresponding to the several cases of a, b, c, being the uppermost side, then

$$(l+a)\tan\alpha + (l+b)\tan\beta + (l+c)\tan\gamma = 0.$$

Let A, B, C, fig. (48), be the triangle. It may be regarded as a lever moveable about a fulcrum at A, and acted on by weights, through the middle points of BC, CA, AB, represented respectively by a, b, c. The weight acting through the middle point of BC may be replaced by two equal forces, each denoted by $\frac{1}{2}a$, acting vertically at B, C. Hence, equating moments about A, we have

$$c . \tfrac{1}{2}c\cos(B+\alpha) + \tfrac{1}{2}a . c\cos(B+\alpha)$$
$$= b . \tfrac{1}{2}b\cos(C-\alpha) + \tfrac{1}{2}a . b\cos(C-\alpha),$$

and therefore

$$\tan\alpha\,\{(b^2+ab)\sin C + (c^2+ac)\sin B\}$$
$$= (c^2+ac)\cos B - (b^2+ab)\cos C:$$

but $$b\sin C = c\sin B:$$

hence, dividing every term of the equation by $b\sin C$ or $c\sin B$, accordingly as the term involves C or B, and putting l for $a+b+c$, we see that

$$(l+a)\tan\alpha = \frac{c+a}{\tan B} - \frac{a+b}{\tan C},$$

an equation which determines the value of α. Similarly we should get

$$(l+b)\tan\beta=\frac{a+b}{\tan C}-\frac{b+c}{\tan A},$$

$$(l+c)\tan\gamma=\frac{b+c}{\tan A}-\frac{c+a}{\tan B}.$$

Adding together these equations, we obtain

$$(l+a)\tan\alpha+(l+b)\tan\beta+(l+c)\tan\gamma=0.$$

(6) Weights of 1 pound and 4 pounds are suspended from the ends of a straight lever without weight: the fulcrum and the point, at which another weight is suspended, dividing the lever into three equal parts: find the magnitude of the third weight, in order that the lever may be in equilibrium.

The required weight is equal to 2 pounds.

(7) A uniform cylinder 2 feet long, weighing 2 pounds, turns about a fulcrum 4 inches distant from one extremity; and from the end nearest to the fulcrum a weight of 16 pounds is suspended: find where a weight of 6 pounds must be suspended in order to produce equilibrium.

The weight must be suspended from the middle point of the lever.

(8) A power P acts vertically downwards at one end of a uniform straight lever, three feet from the fulcrum, the length of the lever being four feet and its weight $\frac{1}{2}P$: what weight will the power balance at the other end of the lever?

The required weight $=\frac{7}{2}P$.

(9) If the weights P and Q, P being the greater, balance on a uniform horizontal cylinder ACB about a fulcrum at C, and the weights be interchanged so that Q acts at A and P at B; shew that the additional weight required at A to maintain equilibrium will be equal to

$$\frac{AB}{AC}\cdot(P-Q).$$

(10) Find the position of equilibrium of a uniform rod $BAOA'B'$, fig. (49), bent at right angles at A, O, A', and moveable about a fixed point at O.

If $OA = 2a$, $AB = 2b$, $OA' = 2a'$, $A'B' = 2b'$, then, θ being the inclination of OA to the horizon,

$$\tan\theta = \frac{a^2 + 2ab - b'^2}{a'^2 + 2a'b' - b^2}.$$

(11) Three uniform rods AB, BC, CD, rigidly connected together so as to form three sides of a square, rest upon a fulcrum at A: find the inclination of AB to the horizon.

If θ = the inclination of AB to the horizon,

$$\tan\theta = \frac{4}{3}.$$

(12) A uniform rod, of weight W, moveable about a hinge at its lower end, presses at its upper end against a smooth wall, three weights, each equal to P, being suspended from points quadrisecting the rod: find the pressure of the rod on the wall, the inclination of the rod to the horizon being α.

The required pressure is equal to

$$\tfrac{1}{2}(3P + W)\cot\alpha.$$

(13) AB, CD, DE, fig. (50), are three equal uniform rods, rigidly connected together at right angles, B being the middle point of CD: A is a smooth hinge: determine the position of equilibrium of the system.

The inclination of AB to the horizontal line AF must be equal to $\tan^{-1}(6)$.

(14) A toasting-fork with three prongs PH, QK, RL, fig. (51), all in a vertical plane, balances about a point C, the axis AQ of the fork being horizontal: find the position of the point C, the thickness of the fork and prongs being uniform.

Let $AQ = a$, $PR = b$, and let the length of each prong be c: then

$$AC = \frac{a^2 + 2ab + 6ac + 3c^2}{2a + 2b + 6c}.$$

(15) Find the position of equilibrium of the toasting-fork, when it is suspended from the end H of the prong PK.

If θ be the inclination of AQ to the horizon,

$$\tan\theta = \frac{a^2 + 2ac + 2bc + 3c^2}{ab + b^2 + 3bc}.$$

(16) Supposing the fork to rest at A and R on a smooth horizontal plane, find the pressure at the point R, the plane of the three prongs being vertical.

If W = the weight of AQ, the required pressure is equal to

$$W.\frac{a^2 + 2ab + 6ac + 3c^2}{2a^2 + \frac{1}{2}b^2}.$$

(17) Find the position of equilibrium of the toasting-fork, supposing one of the outside prongs to be broken off, and the fork to be suspended from A.

If θ = the inclination of AQ to the horizon,

$$\tan\theta = \frac{1}{bc}(a^2 + 2ab + 4ac + 2c^2).$$

(18) The toasting-fork, one of the outside prongs being removed, is fixed in flat contact with a rigid vertical lamina without weight: it is observed that, when a certain point of the lamina is fixed, there is equilibrium for all attitudes of the system: find the position of this point.

The distances of the required point from AQ, PR, are, respectively,

$$\frac{\frac{1}{2}bc}{a+b+2c} \text{ and } \frac{\frac{1}{2}a^2 - c^2}{a+b+2c}.$$

SECT. 3. *Pressure on the axis.*

(1) To find the position of the fulcrum of a straight uniform lever, 12 feet long, when a pound weight at one end keeps the lever at rest and produces a pressure of 6 pounds on the fulcrum.

Let W denote the weight of the lever in pounds and x the distance, in feet, of the fulcrum from the point of suspension of

the pound weight: then, taking moments about the fulcrum, we have

$$1 \times x = W \times (6 - x) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also, the pressure on the fulcrum being equal to the sum of the weights,

$$6 = 1 + W \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we see that

$$W = 5 \text{ pounds}, \quad x = 5 \text{ feet}.$$

(2) A uniform iron rod, a foot of which weighs $1\frac{1}{2}$ lbs., rests on a fulcrum two feet from one end: to find what weight, suspended from that end, will keep it at rest, when the pressure on the fulcrum is 150 lbs.

Let a be the number of feet in the length of the rod, and P the required weight in pounds. Then, observing that the weight of the rod is equal to $\frac{3}{2}a$ pounds, we have, taking moments about the fulcrum,

$$2P = \frac{3}{2}a\left(\frac{1}{2}a - 2\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also, since the pressure on the fulcrum is equal to the weight of the whole system,

$$150 = P + \frac{3}{2}a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we see that

$$300 - 3a = \frac{3}{4}a(a - 4),$$

$$400 - 4a = a^2 - 4a,$$

$$a = 20;$$

and therefore, by (2),

$$P = 120.$$

(3) A weightless rod OB, fig. (52), is moveable about a fixed point O: a weight W is suspended by a fine string, which passes through a ring fixed to the end B of the rod, and of which

the other end is tied to a fixed point A in the same horizontal line with O, OB being equal to OA: to find the position of equilibrium and the pressure on the point O.

The rod is kept at rest about O by two forces, each equal to W, the one acting in the vertical line BC, the other in the direction BA. In order that there may be equilibrium, the resultant of these two forces must act along BO: but this resultant bisects the angle between the two parts of the string; hence $\angle OBC = \angle OBA$. Denoting therefore $\angle OBA$ by θ, we see that $\angle ABC = 2\theta$, and that $\angle BAO = \frac{1}{2}\pi - 2\theta$: but, since $OA = OB$, therefore $\angle BAO = \angle ABO = \theta$:

hence $$\theta = \tfrac{1}{2}\pi - 2\theta, \quad \theta = \frac{\pi}{6}.$$

If R denote the pressure on the point O,

$$R = 2W \cos\theta = W\sqrt{3}.$$

(4) Two forces P and Q, inclined to each other at an angle α, act upon a rigid lamina, which lies upon a smooth table, one point of the lamina being attached to a fixed point of the table: find the pressure on this point.

The required pressure is equal to

$$(P^2 + Q^2 + 2PQ\cos\alpha)^{\frac{1}{2}}.$$

(5) A cylinder of uniform density, 10 feet long, is kept in equilibrium round a fulcrum by means of a pound weight suspended from one end; find the position of the fulcrum and the weight of the cylinder, when the pressure on the fulcrum is 20 pounds.

The distance of the fulcrum from the end at which the pound weight is suspended is 4 feet 9 inches, and the weight of the cylinder is 19 pounds.

CHAPTER VI.

CENTRE OF GRAVITY.

SECT. 1. *Geometrical Method.*

(1) IF G be the centre of gravity of a triangle ABC, three forces in the direction of and proportional to GA, GB, GC, will keep the point at rest.

Complete the parallelogram $AGBD$, fig. (53), and join GD. The diagonals GD, AB, will bisect each other in R, and R, G, C, by the property of the centre of gravity of a triangle, will lie in a straight line.

Now $DG = 2RG = GC$. Hence the three forces GA, GB, GC, are equivalent to the three forces GA, GB, DG, which, by the principle of the parallelogram of forces, will balance each other.

(2) To find the centre of gravity of five equal heavy particles, placed at five of the angular points of a regular hexagon.

Let the heavy particles be placed at B, C, D, E, F, fig. (54). Join AD: join also BF, CE, cutting AD in M, N. Let O be the middle point of AD. Let P be the weight of each particle. Then P, P, at B, F, may be collected at their centre of gravity M: and P, P, at C, E, may be collected at their centre of gravity N. Again $2P$, $2P$, at M, N, may be collected at their centre of gravity O. Thus the particles at B, C, E, F, may be all condensed at O. Again $4P$ at O, and P at D, may be condensed at their centre of gravity G, OG being equal to $\frac{1}{5}OD$.

(3) Heavy particles are placed at the angles of a triangle, their weights being proportional to the sides opposite to them: to prove that their centre of gravity coincides with the centre of the circle inscribed in the triangle.

Let P, Q, R, be the weights of the particles at the angular points A, B, C, fig. (55), respectively. Draw AH, BK, CL, bisecting the angles A, B, C, of the triangle. Then

$$\frac{BH}{CH}=\frac{BA}{CA}, \text{ by Euclid,}$$

$$=\frac{R}{Q}, \text{ by the hypothesis,}$$

$$\text{or } Q.BH=R.CH.$$

Hence the centre of gravity of Q, R, and therefore of P, Q, R, is in the line AH. Similarly it may be shewn to be in the line BK, and in the line CL. Hence it coincides with the intersection of AH, BK, CL, that is, with the centre of the inscribed circle.

(4) To prove that the centre of gravity of the periphery of a triangle cannot coincide with the centre of gravity of the triangular area unless the triangle be equilateral.

Let ABC, fig. (56), be any triangle: bisect the side BC in P and join AP. Let Q, R, be the middle points of the sides AC, AB: join QR, cutting AP in E. Since AB, AC, are bisected in R, Q, the line RQ must be parallel to BC.

The centre of gravity of the triangular area lies in AP: it is plain that, the centre of gravity of BC being at P, the centre of gravity of the periphery cannot lie in this line, and therefore cannot coincide with the centre of gravity of the area, unless the centre of gravity of AB and AC lie in this line: hence we must have

$$AB : AC :: EQ : ER$$

$$:: PC : PB,$$

and therefore, PC being equal to PB, we must have

$$AB=AC.$$

Similarly it may be shewn that the periphery and area cannot have the same centre of gravity unless $AB=BC$: thus the

coincidence of the centres of gravity can take place only in an equilateral triangle.

(5) If lines be drawn from any point whatever to four fixed points in the same plane with it, and these lines represent forces, to shew that their resultant will pass through a certain fixed point and will be proportional to the distance of the first point from it.

Let O, fig. (57), be the variable and A, A', A'', A''', the fixed points: complete the parallelograms $OABA'$, $OBB'A''$, $OB'B''A'''$, and join the diagonals AA', BA'', $B'A'''$; C, C', C'', being the points of intersection of the diagonals in the several parallelograms. Bisect OC'' in D.

Then the four forces OA, OA', OA'', OA''', are equivalent to OB, OA'', OA''', and therefore to OB', OA''', and therefore to a single resultant OB''.

Again, place four weights, each equal to P, at A, A', A'', A'''. The centre of gravity of P, P, at A, A', coincides with that of P, P, at O, B: the centre of gravity of P, P, P, at A, A', A'', coincides with that of P, P, P, at O, B, A'', or of P, $2P$, at O, C', or of $2P$, P, at O, B': the centre of gravity of P, P, P, P, at A, A', A'', A''', coincides with that of $2P$, P, P, at O, B', A''', or of $2P$, $2P$, at O, C'', or of $4P$ at D. Thus we see that the resultant OB'' passes through the centre of gravity D of four equal weights, placed at the points A, A', A'', A''', and is represented by OB'', which is equal to four times OD or varies as OD.

(6) ABC, fig. (58), is a triangle; α, β, γ, are the middle points of the sides opposite to A, B, C, respectively: $\alpha\beta$, $\alpha\gamma$, are joined and the figure $A\beta\alpha\gamma$ is removed from the triangle: to determine the position of the centre of gravity of the remainder.

Draw $\beta\mu$, $\gamma\nu$, parallel to $A\alpha$: these lines will bisect $C\alpha$, $B\alpha$, respectively, and, if μ', ν', be taken in $\beta\mu$, $\gamma\nu$, such points that $\mu\mu' = \frac{1}{3}\mu\beta$ and $\nu\nu' = \frac{1}{3}\nu\gamma$, μ', ν', will be the centres of gravity of the triangles $C\alpha\beta$, $B\alpha\gamma$, for it is easily seen that $C\alpha$, $B\alpha$, are bisected in μ, ν, respectively. But $\mu\beta = \frac{1}{2}A\alpha = \nu\gamma$: hence $\mu\mu' = \nu\nu'$,

and therefore $\mu'\nu'$ is parallel to CB. Moreover, the triangles $Ba\gamma$, $Ca\beta$, are equal: hence the centre of gravity of the two triangles will lie in the middle point between μ' and ν', that is, in the intersection of $\mu'\nu'$ with $A\alpha$. Hence the required centre of gravity lies in $A\alpha$ at a distance from α equal to $\frac{1}{3}A\alpha$.

(7) To prove the following geometrical construction for the centre of gravity of any quadrilateral. Let E be the intersection of the diagonals, and F the middle point of the line which joins their middle points: draw the line EF and produce it to G, making FG equal to one third of EF: then G shall be the centre of gravity required.

Let K, H, fig. (59), be the middle points of the diagonals AC, BD. Join KH, AH, BK, CH, DK. Take $PK = \frac{1}{3}BK$, $QK = \frac{1}{3}DK$, $HS = \frac{1}{3}AH$, $HR = \frac{1}{3}CH$. Join PQ, RS, cutting AC, BD, in M, N, respectively; join MN. Let G be the intersection of PQ, RS; join EG cutting MN, HK, in O, F, respectively.

Since the centres of gravity of the triangles ABC, ADC, are at P, Q, respectively, the centre of gravity of the whole quadrilateral must be in the line PQ: by like reasoning, it must also be in the line RS: hence it must be at G, the intersection of these two lines.

Again, since $PK : QK :: BK : DK$, PQ is parallel to BD: similarly, RS is parallel to CA: hence $EMGN$ is a parallelogram.

Again $\quad EM : EK :: BP : BK :: 2 : 3,$

and $\quad EN : EH :: AS : AH :: 2 : 3;$

and therefore

$$EM : EK :: EN : EH,$$

and consequently MN is parallel to KH: hence, MN being bisected in O, HK is bisected in F.

Again $\quad EO : EF :: EM : EK :: 2 : 3,$

and therefore $\quad EO = \frac{2}{3}EF$, and $FO = \frac{1}{3}EF$.

Hence
$$\begin{aligned} FG &= GO - FO \\ &= EO - FO \\ &= \tfrac{2}{3}EF - \tfrac{1}{3}EF = \tfrac{1}{3}EF. \end{aligned}$$

(8) How long a piece must be cut off from one end of a rod, of length $2a$, in order that the centre of gravity of the rod may approach towards the other end through a distance b?

The length of the piece must be $2b$.

(9) If a quadrilateral be divided by one of its diagonals into two equal triangles, shew that it will balance about that diagonal.

(10) If the sides of a triangle be taken, two and two, to represent forces, acting in each case from the angle made by the sides, prove that each of the three pairs will balance about the centre of gravity of the triangle.

(11) If the lengths of the sides of a right-angled triangle be 3, 4, 5 feet, find the distance of the centre of gravity from each side.

The distances of the centre of gravity from the three sides are $\frac{4}{3}$, 1, $\frac{4}{5}$, feet, respectively.

(12) A square and a rectangle of uniform density are joined together in one plane at a common side: find the length of the rectangle in order that the two may balance about that side, the thickness of the square being double that of the rectangle.

The length of the rectangle must be equal to a diagonal of the square.

(13) If ABC be an isosceles triangle, having a right angle at C, and D, E, be the middle points of AC, AB, respectively, prove that a perpendicular from E upon BD will pass through the centre of gravity of the triangle BDC.

(14) If the centre of gravity of a four-sided figure coincide with one of its angular points, shew that the distances of this point and the opposite angular point from the line joining the other two angular points are as 1 to 2.

(15) Draw a line through one angle of a square area, cutting off a triangle, so that the remaining quadrilateral, when suspended from the obtuse angle, may rest with one side vertical.

Upon the side BC of the square construct the equilateral triangle BMC, fig. (60): draw MV at right angles to BC: take MN equal to half a side of the square: draw NO, parallel to VB, cutting AB in O: join DO. Then DO is the required line drawn through the angle D of the square, and cutting off the required triangle ADO.

(16) If ABC be a triangular board, having B an obtuse angle and AB less than BC; prove that the board may stand, when AB is placed on a horizontal plane, and will certainly stand when BC is so placed.

(17) A plane quadrilateral $ABCD$ is bisected by the diagonal AC, and the other diagonal divides AC into two parts in the ratio $p : q$; shew that the centre of gravity of the quadrilateral lies in AC and divides it into two parts in the ratio $2p + q : 2q + p$.

(18) O is any point within the area of a triangle ABC: another triangle is formed by joining the centres of gravity G, H, K, of the triangles BOC, COA, AOB: prove that the area of the triangle GHK is one ninth of the area of the original triangle and similar to it.

(19) A uniform wire is bent into the form of three sides AB, BC, CD, of an equilateral polygon, and its centre of gravity is at the intersection of AC, BD: prove that the polygon must be a regular hexagon.

(20) $ABCD$ is a plane quadrilateral figure and a, b, c, d, are respectively the centres of gravity of the triangles BCD, CDA, DAB, ABC: prove that the quadrilateral $abcd$ is similar to the quadrilateral $ABCD$.

(21) Prove that the straight lines joining the middle points of opposite edges of a tetrahedron intersect and bisect each other.

SECT. 2. *Co-ordinate Method.*

Let P, P', P'',... be any number of weights in one plane, and (x, y), (x', y'), (x'', y''),... their co-ordinates referred to any axes, rectangular or oblique, in this plane. Then, $\bar{x}$, $\bar{y}$, being the co-ordinates of the centre of gravity of the weights,

$$\bar{x} = \frac{Px + P'x' + P''x'' + \dots}{P + P' + P'' + \dots},$$

$$\bar{y} = \frac{Py + P'y' + P''y'' + \dots}{P + P' + P'' + \dots}.$$

These formulæ may be written briefly thus:

$$\bar{x} = \frac{\Sigma(Px)}{\Sigma(P)}, \quad \bar{y} = \frac{\Sigma(Py)}{\Sigma(P)}.$$

If all the weights lie in one line, then, this line being taken for the axis of x, the centre of gravity will also lie in this line, and its position will be given by the single formula

$$\bar{x} = \frac{\Sigma(Px)}{\Sigma(P)}.$$

If the weights be referred to any three axes in space,

$$\bar{x} = \frac{\Sigma(Px)}{\Sigma(P)}, \quad \bar{y} = \frac{\Sigma(Py)}{\Sigma(P)}, \quad \bar{z} = \frac{\Sigma(Pz)}{\Sigma(P)}.$$

(1) A uniform board is composed of a square $ABCD$, fig. (61), and an equilateral triangle AEB: to find the distance of the centre of gravity of the whole board from the point C or D.

Bisect CD in O, and join EO: the centre of gravity of the whole board will evidently lie in EO.

Let $a =$ the length of a side of the square: then the area of the square is a^2 and the distance of its centre of gravity from O is $\frac{1}{2}a$. Also, the area of the triangle AEB is $\frac{1}{4}a^2\sqrt{3}$, and the

distance of its centre of gravity from AB is $\frac{a}{2\sqrt{3}}$. Hence, G being the centre of gravity of the whole board,

$$OG = \frac{a^2 \cdot \frac{a}{2} + \frac{a^2\sqrt{3}}{4} \cdot \left(a + \frac{a}{2\sqrt{3}}\right)}{a^2 + \frac{1}{4}a^2\sqrt{3}}$$

$$= a \cdot \frac{5 + 2\sqrt{3}}{8 + 2\sqrt{3}}.$$

Hence, $$CG^2 = a^2\left\{\frac{1}{4} + \left(\frac{5 + 2\sqrt{3}}{8 + 2\sqrt{3}}\right)^2\right\} = a^2 \cdot \frac{14 + 7\sqrt{3}}{19 + 8\sqrt{3}},$$

and therefore $$CG = a\sqrt{\frac{7}{2}} \cdot \frac{1 + \sqrt{3}}{4 + \sqrt{3}}.$$

(2) To find the centre of gravity of the solid included between two right cones on the same base, the vertex of one cone being within the other; and to determine its limiting position if the vertices approach to coincidence.

Let h, h', be the altitudes of the two cones, and let x be the distance of the centre of gravity of the solid from the common base: then, since the volumes of the cones are as their altitudes, and since the distance of the centre of gravity of a cone from its base is equal to a quarter of its altitude, we have

$$x(h - h') + \tfrac{1}{4}h' \cdot h' = \tfrac{1}{4}h \cdot h,$$

whence $$x = \frac{1}{4}\frac{h^2 - h'^2}{h - h'}$$

$$= \tfrac{1}{4}(h + h');$$

and, when the vertices of the cones approach to coincidence, h' approaches h as its limit, and therefore, ultimately, x is equal to $\frac{1}{2}h$.

(3) A beetle crawls from one end A of a straight fixed rod to the other end B: to find the consequent alteration in the position of the centre of gravity of the rod and beetle.

Let $a =$ the length and $P =$ the weight of the rod, and $W =$ the weight of the beetle.

Then the original distance of the centre of gravity of the rod and beetle from A is equal to

$$\frac{\frac{1}{2}aP}{P+W},$$

and its distance from A, when the beetle arrives at B, is equal to

$$\frac{\frac{1}{2}aP+aW}{P+W}:$$

hence the centre of gravity has moved through a space equal to

$$\frac{Wa}{P+W}.$$

(4) A circular piece is cut out of a rectangular board $ABCD$, fig. (62), the two sides AB, AD, touching the circumference of the circle: to find the position of the centre of gravity of the remaining portion.

Produce AB, AD, indefinitely to x, y, Ax, Ay, being taken as axes of co-ordinates. Let $AB=a$, $AD=b$, $r =$ the radius of the circle. Let $\bar{x}$, $\bar{y}$, be the co-ordinates of the centre of gravity of the remaining portion of the board. Then

$$ab\,.\,\tfrac{1}{2}a = \pi r^2\,.\,r + (ab - \pi r^2)\,\bar{x},$$

and therefore $$\bar{x} = \frac{\frac{1}{2}a^2b - \pi r^3}{ab - \pi r^2}.$$

Similarly $$\bar{y} = \frac{\frac{1}{2}b^2a - \pi r^3}{ba - \pi r^2}.$$

(5) A weight P is supported on a smooth inclined plane by a string, parallel to the plane, which passes over a fixed pully, and is attached to a weight Q; to prove that, when Q is moved vertically, the centre of gravity of P and Q will neither rise nor fall.

Let x be the distance of P and y the distance of Q from the

pully: then the depth of the centre of gravity of P and Q below the pully is equal to

$$\frac{Px \sin \alpha + Qy}{P + Q}.$$

But, since P is at rest, we have, resolving the forces, which act upon it, parallel to the plane,

$$Q = P \sin \alpha.$$

Hence, c being the length of the string, the depth of the centre of gravity is equal to

$$\frac{Q(x+y)}{P+Q} = \frac{Qc}{P+Q},$$

which is independent of the depth of Q below the pully.

(6) The inscribed circle being cut out of a right-angled triangle, the sides of which are 3, 4, 5; to find the centre of gravity of the remainder.

Let $OA = 3$, $OB = 4$, and therefore $AB = 5$. Let OAx, OBy, fig. (63), be taken as the co-ordinate axes.

Since the centre of gravity of the circular area and the remainder of the triangle must coincide with the centre of gravity of the triangle, we have, $\bar{x}$, $\bar{y}$, being the co-ordinates of the required centre of gravity,

$\frac{1}{3} OA \times$ area of triangle = rad. of circle $\times$ its area

$+ \bar{x} \times$ area of remainder of triangle,

and

$\frac{1}{3} OB \times$ area of triangle = rad. of circle $\times$ its area

$+ \bar{y} \times$ area of remainder of triangle.

But, r denoting the radius of the circle,

$$r(3 + 4 + 5) = 3 \times 4,$$

and therefore, $r = 1$.

Our two equations accordingly become

$$6 = \pi + (6 - \pi)\bar{x}, \quad \bar{x} = 1;$$

and

$$\frac{4}{3}\times 6 = \pi + (6-\pi)\,\bar{y},\quad \bar{y} = \frac{8-\pi}{6-\pi}.$$

The value of $\bar{x}$ shews that the required centre of gravity lies in a line through the centre of the circle parallel to OB.

(7) Three weights, placed at the corners of a triangle, are proportional to the opposite sides; to shew that their centre of gravity coincides with the centre of the circle inscribed in the triangle.

Let a, b, c, be the sides of the triangle, and λa, λb, λc, the weights at the opposite angles: let $\bar{x}$, $\bar{y}$, $\bar{z}$, denote the distances of the centre of gravity of the weights from the sides a, b, c, respectively: then, p denoting the perpendicular upon the side a from the opposite angle,

$$\bar{x} = \frac{\lambda a \,.\, p}{\lambda a + \lambda b + \lambda c}$$

$$= \frac{2A}{a+b+c},$$

where A is the area of the triangle. Similarly

$$\bar{y} = \frac{2A}{a+b+c},\quad \bar{z} = \frac{2A}{a+b+c}.$$

These results shew that the distances of the centre of gravity from the three sides are all equal to the radius of the inscribed circle. The centre of gravity of the weights must therefore coincide with the centre of the circle.

(8) To find the centre of gravity of the periphery of a triangle formed by a piece of uniform wire.

Let ABC, fig. (64), be the triangle. Bisect the sides BC, CA, AB, in α, β, γ, respectively, and join $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

The centre of gravity of the triangle ABC will be the same as that of three weights in the proportion of BC, CA, AB, placed respectively at α, β, γ: but BC, CA, AB, are proportional to $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$; hence the centre of gravity of ABC coincides with

that of three weights, proportional to $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, placed at the angular points α, β, γ, of the triangle $\alpha\beta\gamma$. Hence, by the preceding theorem, the centre of gravity of ABC coincides with the centre of the circle inscribed in the triangle $\alpha\beta\gamma$.

(9) One corner of a triangle, equal to $\frac{1}{n}$th part of its area, is cut off by a line parallel to its base: to find the centre of gravity of the remainder.

Let $B'C'$, fig. (65), be a line parallel to the base BC of the triangle ABC, the area $AB'C'$ being one n^{th} of that of ABC. Bisect BC in P and join AP, which will bisect $B'C'$ in P'. Let u denote the area of ABC, and $\bar{x}$ the distance of G, the centre of gravity of $BCC'B'$, from A. Then

$$\frac{2}{3}.AP'.\frac{u}{n}+\bar{x}\left(u-\frac{u}{n}\right)=\frac{2}{3}AP.u,$$

$$AP'+\frac{3}{2}\bar{x}(n-1)=n.AP.$$

But, the triangles $AB'C'$, ABC, being similar,

$$u:\frac{u}{n}::AP^2:AP'^2,$$

and therefore $$AP'=\frac{AP}{\sqrt{n}}.$$

Hence $$\frac{3}{2}(n-1)\bar{x}=AP\left(n-\frac{1}{\sqrt{n}}\right),$$

$$\bar{x}=\frac{2}{3}AP.\frac{n^{\frac{3}{2}}-1}{n^{\frac{1}{2}}.(n-1)}=\frac{2}{3}AP.\frac{n+n^{\frac{1}{2}}+1}{n^{\frac{1}{2}}(n^{\frac{1}{2}}+1)}.$$

(10) A piece of uniform wire is formed into a triangle; to find the distance of the centre of gravity of the periphery of the triangle from each of the sides; and to shew that, if x, y, z, be the three distances, and r the radius of the inscribed circle, then

$$4xyz-r^2(x+y+z)=r^3.$$

Let a, b, c, be the sides of the triangle. Then, p denoting the distance between the side a and the opposite angle,

$$x = \frac{\frac{1}{2}p \, . \, b + \frac{1}{2}p \, . \, c}{a + b + c}:$$

but $\qquad pa = \text{twice the area of the triangle}$

$$= r(a + b + c);$$

hence $$x = \frac{r}{2a}(b + c):$$

similarly $$y = \frac{r}{2b}(c + a),$$

and $$z = \frac{r}{2c}(a + b).$$

Hence $$4xyz - r^2(x + y + z)$$

$$= \frac{r^3}{2abc}\{(b + c)(c + a)(a + b) - bc(b + c) - ca(c + a) - ab(a + b)\}$$

$$= r^3.$$

(11) AOB, fig. (66), is a bent lever of uniform thickness; to find the distance of its centre of gravity from O.

Let $OA = 2a$, $OB = 2b$, $\angle AOB = \omega$. Let G be the centre of gravity of the lever: draw GH parallel to yBO, and join OG: let $OH = \bar{x}$, $GH = \bar{y}$. The weights of OA, OB, supposed to be collected at their centres of gravity, may be represented by λa, λb. Hence

$$\bar{x} = \frac{\lambda a \, . \, a}{\lambda a + \lambda b} = \frac{a^2}{a + b},$$

and $$\bar{y} = \frac{\lambda b \, . \, b}{\lambda a + \lambda b} = \frac{b^2}{a + b}.$$

Hence $$OG^2 = \bar{x}^2 + \bar{y}^2 + 2\bar{x}\bar{y}\cos\omega,$$

and therefore

$$OG = \frac{(a^4 + b^4 + 2a^2b^2\cos\omega)^{\frac{1}{2}}}{a + b}.$$

(12) Three weights, of 2, 3, 4, ounces respectively, lie in a straight line; the distance between the first and second is 10 inches, that between the second and third is 5 inches: find the centre of gravity of the three weights.

The centre of gravity coincides with the place of the middle weight.

(13) Four weights, of 1, 2, 3, 4, pounds, are placed in order, at equal distances, one inch asunder, on a thin rigid rod: find the point on which they will balance.

The required point is at the third weight.

(14) If, at the two ends and at distances of 12 inches from the ends of a rod, a yard long, four weights, of 3, 5, 7, 9, pounds, be fixed in order: shew that their centre of gravity is at a distance of one inch from the third weight.

(15) At the corners of a square, taken in order, are placed weights 1, 3, 5, 7: find their centre of gravity.

If a be the length of a side of the square, the distance of the centre of gravity from the side (1, 7) is $\frac{a}{2}$, and from the side (5, 7) is $\frac{a}{4}$.

(16) A given number of equal weights are suspended from a horizontal line, and a given quantity of string is to be used: determine the distance of the centre of gravity of the weights from the line.

If n be the number of the weights, and l the whole length of string used, the required distance is equal to $\frac{l}{n}$.

(17) Find the distance of the centre of gravity of three equal weights, placed at the angles of a right-angled triangle, from the right angle.

The required distance is equal to one-third of the hypotenuse.

(18) Find the distance of the centre of gravity of a right-angled isosceles triangle and the squares described on the two equal sides, from the vertex of the triangle.

If $a =$ the length of either of the equal sides, the required distance is equal to

$$\frac{a\sqrt{2}}{15}.$$

(19) The circumference of a circle is divided into three parts, which are as 1, 2, 3: compare the distances of their centres of gravity from the centre of the circle.

These distances are respectively in the proportion

$$6 : 3\sqrt{3} : 4.$$

(20) Prove that the excess of the altitude of a man's centre of gravity from the ground, when he stands with his legs in contact, above its altitude, when he stands with his legs asunder, varies as the square of the sine of a quarter of the angle between his legs.

(21) A beetle crawls from one end A to the other end B of a fixed bent rod: determine the length of the space described by the centre of gravity of the rod and beetle.

If c be the rectilinear distance of A from B, and W, W', be the respective weights of the beetle and rod, the required space is equal to

$$\frac{Wc}{W+W'}.$$

(22) Two uniform boards, one square and the other circular, are placed in one plane, so that they touch each other only at the middle point of a side of the square: find the ratio of a side of the square to the radius of the circle, in order that the distance of their centre of gravity from the centre of the square may be equal to the radius of the circle.

The required ratio is equal to $\frac{1}{2}\pi$.

(23) A side of a square board is a tangent at its middle point to the circumference of a circular board, the plane of the

square being at right angles to that of the circle: find the ratio of a side of the square to the radius of the circle, in order that the centre of gravity of the boards may be equidistant from both.

The required ratio is equal to $(2\pi)^{\frac{1}{3}}$.

(24) From an isosceles triangular lamina ABC, of which the sides AB, BC, are equal, an isosceles portion APC is cut away, AP, CP, being equal: find the centre of gravity of the remainder.

If h, h', be the distances of B, P, respectively, from AC, the distance of the required centre of gravity from AC is equal to

$$\tfrac{1}{3}(h+h').$$

(25) An infinite number of particles are placed in a straight line, at equal distances from each other, their weights forming a decreasing geometrical series: find the distance of their centre of gravity from the largest weight.

If a denote the interval between two successive weights, and r the common ratio, the required distance is equal to

$$\frac{ar}{1-r}.$$

(26) In a trapezium $ABCD$, if AB be parallel to CD, and E, F, be the middle points of AB, CD, respectively, prove that, a, b, being the lengths of AB, CD, the centre of gravity of the trapezium divides EF in the ratio $a+2b : 2a+b$.

(27) The centres of two circles, which touch each other internally, are made to approach indefinitely near to each other: find the ultimate position of the centre of gravity of the area included between the circumferences of the circles.

The ultimate position of the centre of gravity will bisect the radius of that point in the circumference which is most distant from the point of contact.

(28) Prove that the centre of gravity of four equal particles in any position is the same as that of four other equal particles,

each of which is placed at the centre of gravity of three of the former.

(29) If two weights support each other on inclined planes by means of a fine string passing over the common vertex of the two planes, and the system is set in motion, prove that the centre of gravity of the weights moves in a horizontal line.

(30) A weight of given magnitude moves along the circumference of a circle, in which are fixed also two other weights: prove that the locus of the centre of gravity of the three weights is a circle. If the immoveable weights be varied in magnitude, their sum being constant, prove that the corresponding circular loci intercept equal portions of the chord joining the two immoveable weights.

(31) If p_1, p_2, p_3, be the perpendicular distances, from the sides, of the centre of gravity of the portion of a triangle remaining, after the part within the inscribed circle is removed; prove that

$$\frac{1}{p_1+u}+\frac{1}{p_2+u}+\frac{1}{p_3+u}=\frac{3\pi r}{su},$$

where $r=$ the radius of the inscribed circle, $s=$ the semi-perimeter of the triangle, and $u=\dfrac{\pi r^2}{s-\pi r}$.

(32) Find the centre of gravity of a straight rod, the density of which increases in a given proportion with the distance from one of its extremities.

The distance of the centre of gravity from the extremity is equal to two-thirds of the length of the whole rod.

(33) n cylinders of the same height h, the radii of which are equal to r_1, r_2, r_3, ... r_n, respectively, stand upon one another with their axes in the same straight line; prove that, if x be the distance of their common centre of gravity from the base of the first, then

$$x=\frac{1}{2}h\cdot\frac{2r_1^2+3r_2^2+5r_3^2+\ldots+(2n-1)r_n^2}{r_1^2+r_2^2+r_3^2+\ldots+r_n^2}.$$

(34) Assuming that

$$1.1+2.3+3.5+\ldots+n.(2n-1)=\frac{1}{6}n(n+1)(4n-1),$$

find the centre of gravity of a frustum of a paraboloid of revolution cut off by a plane perpendicular to its axis.

If $a=$ the length of the axis of the frustum, the distance of the centre of gravity from the vertex is equal to $\frac{2}{3}a$.

(35) If a uniform wire be bent into the form of a triangle and at points in the sides there be placed three beads, the weights of which are proportional to the sides on which they are; prove that, when the beads are moved with equal velocities in the same direction along the sides, there will be no change in the position of the centre of gravity of the whole system.

(36) Three uniform rods OA, OB, OC, have a common end O: find the distance of their centre of gravity from O.

If $OA=a$, $OB=b$, $OC=c$, $\angle BOC=\alpha$, $\angle COA=\beta$, $\angle AOB=\gamma$, and $d=$ the required distance,

$$d^2=a^4+b^4+c^4+2b^2c^2\cos\alpha+2c^2a^2\cos\beta+2a^2b^2\cos\gamma.$$

(37) Shew that forces, acting at the centre of gravity of a system of equal heavy particles in space and represented in magnitude and direction by straight lines drawn from it to all the particles, will be in equilibrium.

(38) If particles of unequal weights be placed at the angular points of a triangular pyramid, and G_1, G_2, G_3, &c., be their common centres of gravity, respectively, for every possible arrangement of the particles; prove that the centre of gravity of equal particles, placed at G_1, G_2, G_3, &c., is the centre of gravity of the pyramid.

SECT. 3. *Upsetting.*

(1) If a given uniform right-angled triangular lamina ABC, fig. (67), in which C is the right angle, be placed in a vertical position on an inclined plane with its base BC in contact with the plane, sliding being prevented by a small peg at C: to find the inclination of the plane in order that the lamina may just not topple over.

Supposing the lamina to be just on the point of toppling over, the centre of gravity must be vertically above C: hence the line CO, which joins C and the middle point O of AB, must be vertical.

Hence the inclination of the plane must be equal to the complement of the angle OCB, and therefore to the angle ACO: but, since a circle may be described about the right-angled triangle ACB with O as a centre, $CO = AO$, and therefore $\angle ACO = \angle BAC$: hence the inclination of the plane must be equal to the angle BAC of the lamina.

(2) A cubical box is half filled with water and placed upon a rough rectangular board, so as to have the edges of its base parallel to those of the rectangle; if the board be slowly inclined to the horizon, determine whether the box will slide down or topple over.

The centre of gravity of the water and box would evidently be vertically above the lower edge of the box if the board were inclined to the horizon at an angle of 45°, and it is easily seen that such would not be the case for any smaller inclination. Now the box will or will not slide before the inclination of the board reaches 45° accordingly as μ, the coefficient of friction, is less or greater than unity. Thus the box will slide down or topple over, accordingly as μ is less or greater than unity.

(3) An isosceles triangle is placed with its base upon a rough inclined plane, the coefficient of friction between the triangle and the plane being given; to find the vertical angle of the triangle, supposing that, on a gradual increase of the inclination of the plane, the triangle assumes simultaneously a rolling and a sliding motion.

Let $\alpha =$ the inclination of the plane AB, fig. (68), upon which the triangle CDE is placed, and let $\angle CED = \beta$. Draw the perpendicular EF, and let G be the centre of gravity of the triangle: join CG. Then, supposing the triangle to be on the point of sliding, we know that

$$\tan \alpha = \mu.$$

Again, if the triangle be at the same time on the point of rotating about C, it is plain that CG is vertical: hence

$$\tan \frac{\beta}{2} = \frac{CF}{EF} = \frac{CF}{3FG} = \tfrac{1}{3} \tan \alpha = \frac{\mu}{3},$$

and thus the required vertical angle is equal to

$$2 \tan^{-1} \left(\frac{\mu}{3}\right).$$

(4) An equilateral triangle is placed upon an inclined plane, its lowest angle being fixed: find how high the plane may be elevated before the triangle rolls.

The greatest elevation is an angle of 60^0.

(5) If a round table stand on three legs, placed on the circumference at equal distances, shew that a body in weight equal to but not greater than the table may be placed on any part of it without risk of upsetting it.

(6) A rough rectangular board is placed with one edge upon a rough inclined plane, the plane of the board being vertical: find the condition that, as the plane is gradually elevated, a motion of sliding and of rolling may commence simultaneously.

If a be the length of the edge which is in contact with the inclined plane, b the length of an edge perpendicular to the plane, and μ the coefficient of friction, the required condition is

$$\mu = \frac{a}{b}.$$

(7) A solid cone is placed with its base in contact with a plane, inclined at an angle of 30^0 to the horizon, and is prevented

sliding by a small obstacle at the lowest point of its base: determine the height of the cone, in terms of the radius of the base, in order that the cone may be just on the point of upsetting.

If $r =$ the radius of the base, the height of the cone must be equal to $4r\sqrt{3}$.

SECT. 4. *Body supported at a point.*

If a body or system of bodies, supported at a point, be at rest, the centre of gravity is vertically above or vertically below the point.

(1). If AB, BC, fig. (69), be two uniform rods, rigidly connected together at B, and EF be a line joining their middle points, and D be taken in EF so that $DF : DE :: AB : BC$; to shew that they will balance on A, if DA be vertical, or on C, if DC be vertical.

The centres of gravity of AB, BC, are at their middle points E, F, respectively: let P, Q, be their respective weights: then

$$DF : DE :: AB : BC$$
$$:: P : Q,$$

and therefore the centre of gravity of the system ABC is at D: hence the truth of the proposition.

(2) A material triangular lamina hangs at rest, one of its angles being supported at a fixed point: to find the angle which the lower side makes with the horizon.

Let A, fig. (70), be the point of suspension: bisect BC in H and join AH: then, since the lamina is at rest, the line AH, which contains its centre of gravity, must be vertical. Produce BC to meet the horizontal line through A in the point D: let $\angle ADB = \theta$.

Then, from the triangles BAH, CAH, we have

$$\frac{BH}{AH} = \frac{\sin \angle BAH}{\sin B} = \frac{\cos (B+\theta)}{\sin B},$$

and $$\frac{CH}{AH} = \frac{\sin \angle CAH}{\sin C} = \frac{\cos (C - \theta)}{\sin C}:$$

hence, BH being equal to CH,

$$\frac{\cos (B + \theta)}{\sin B} = \frac{\cos (C - \theta)}{\sin C},$$

$$\cot B - \tan \theta = \cot C + \tan \theta,$$

and therefore $$\tan \theta = \tfrac{1}{2} (\cot B - \cot C).$$

(3) A right-angled triangular board hangs at rest from the right angle, and the hypotenuse is inclined at sixty degrees to the plumb line: compare the lengths of the sides.

The length of one side is to that of the other as $\sqrt{3}$ to 1.

(4) A heavy body is supported in a given position by means of a string, which is fastened to two given points in the body, and passes over a smooth peg: find the length of the string.

Let A, B, be the two points in the body, and G its centre of gravity: let a, b, denote the lengths AG, BG, and α, β, the inclinations of AG, BG, to the vertical: then the length of the string is equal to

$$\frac{a \sin \alpha + b \sin \beta}{a \sin \alpha - b \sin \beta} \cdot \{a^2 + b^2 - 2ab \cos (\alpha - \beta)\}^{\frac{1}{2}}.$$

(5) A fine string $ACBP$, tied to the end A of a uniform rod AB of weight W, passes through a fixed ring at C, and also through a ring at the end B of the rod, the free end of the string supporting a weight P; if the system be in equilibrium, prove that

$$AC : BC :: 2P + W : W.$$

(6) A piece of uniform wire is bent into three sides of a square $ABCD$, of which the side AD is wanting; shew that, if it be hung up by the two points A and B successively, the angle between the two positions of BC is $\tan^{-1} 18$.

(7) A triangular lamina ABC, having a right angle C, is suspended from the angle A, and the side AC makes an angle α with the vertical; it is then suspended from B, and the side BC makes an angle β with the vertical: prove that

$$\cot\alpha \,.\, \cot\beta = 4.$$

(8) A lamina of any form is suspended first from a point A and then from a point B, A and B being points in the lamina: it is found that the angles, which the straight line AB makes with the horizon in these two positions, are α and β respectively: prove that the distance of the centre of gravity of the lamina from AB is equal to

$$\frac{AB}{\tan\alpha + \tan\beta}.$$

(9) If a triangular lamina be suspended from a fixed point by strings attached to its angles, shew that the tensions of the strings are proportional to their respective lengths.

CHAPTER VII.

EQUILIBRIUM OF A BODY ACTED ON BY THREE FORCES.

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(1) A PAIR of compasses, the hinge of which is so rusty as to prevent motion, rests upon a smooth vertical circular board: to find the pressure on the board.

The compasses are acted on by three forces, viz. the reactions of the board, and the weight of the compasses, which may be regarded as collected at their centre of gravity. Now the directions of the reactions pass through the centre of the board, and therefore the direction of the weight must also do so: but the centre of gravity of the compasses lies in the line joining their hinge and the centre of the board. Hence the hinge must be vertically above the centre of the board.

Since then the compasses must rest symmetrically, the reactions must be equal: let R denote each of the reactions, and W the weight of the compasses. Let 2α represent the angle between the legs of the compasses. Then, resolving vertically the three forces which act through the centre of the board, we have

$$2R \sin \alpha = W,$$

$$R = \tfrac{1}{2} W \operatorname{cosec} \alpha.$$

(2) A uniform heavy rod, of given length, is to be supported in a given position, its upper end resting at a given point against a smooth vertical wall, by means of a fine string attached to the lower end of the rod and to a point in the wall: to find by a geometrical construction the point in the wall to which the string must be attached.

Let AB, fig. (71), be the rod, resting against the wall at the point B. Let C be the middle point of the rod. Draw BD

horizontally and CD vertically, so as to intersect in the point D. Now the action of the wall on the rod and the weight of the rod are both directed through the point D: hence the tension of the string must also be directed through this point. Join therefore AD and produce it to meet the wall in E: then E is the required point.

(3) To find the position of equilibrium of a uniform rod, the ends of which rest upon two inclined planes which are at right angles to each other.

Let AC, BC, fig. (72), be the two inclined planes, and PQ the position of the rod. In order that the rod may be in equilibrium, its weight, supposed to be collected at its middle point O, must lie vertically below D, the intersection of the normal reactions of the planes at P and Q: thus the semidiagonal OD of the rectangle $PCQD$ and therefore the whole diagonal CD must be vertical. We may therefore give the following construction for the position of equilibrium of the rod. Draw CD vertical and equal to the length of the rod: from D draw DP, DQ, at right angles to the planes: join PQ. Then PQ, being equal to CD, will be the required position of the rod.

(4) A uniform straight rod, moveable about its lower extremity, leans against a vertical wall, and makes an angle of 45° with the horizon: to shew that the pressure against the wall is half the weight of the rod.

Let AB, fig. (73), be the rod, AC a horizontal line meeting the wall BC in C. Draw BO, at right angles to the wall, to meet GO, drawn vertically through the middle point G of the rod, and join AO: produce OG to meet AC in H. Then the weight of the rod, supposed to be collected at G, the reaction of the wall, and the pressure exerted on the rod at A, are represented by OH, HA, AO, respectively: but $OH = BC = 2GH$, and, since $\angle BAC = 45^\circ$, $GH = HA$: hence $OH = 2HA$, or the reaction of the wall, which is equal and opposite to the pressure on the wall, is equal to half the weight of the rod.

(5) The higher end B of a rod AB, fig. (74), moveable about a hinge at its lower end A, is connected by a fine string

with a fixed point C in the horizontal plane through A: to find the pressure on the hinge and the tension of the string.

Let W = the weight of the rod, T = the tension of the string, and R = the reaction of the hinge against the rod. Let G be the centre of gravity of the rod. Draw GH vertically to intersect BC in H: join AH and produce it to meet BD, drawn vertically, in the point D.

Since the directions of W and T both pass through H, it follows that R's direction must also pass through H. Hence AB is kept at rest by W parallel to DB, T along BH, and R along HD: hence

$$R = W \cdot \frac{HD}{BD},$$

and
$$T = W \cdot \frac{HB}{BD}.$$

(6) A uniform heavy rod is placed across a smooth horizontal rail, and rests with one end against a smooth vertical wall, the distance of which from the rail is $\left(\frac{1}{16}\right)^{\text{th}}$ of the length of the rod: to find the position of equilibrium.

The reaction of the wall, the reaction of the rail, and the weight of the rod, must act along lines which pass through a single point: let AC, CO, GC, fig. (75), be the respective lines of action.

Draw OH at right angles to the wall.

Let $AG = a = BG$, and therefore $OH = \frac{1}{8}a$.

Then, θ being the inclination of AB to the vertical,

$$OH = OA \cdot \sin\theta = AC \cdot \sin^2\theta = AG \cdot \sin^3\theta,$$

$$a \sin^3\theta = \frac{a}{8}, \quad \sin\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{6}.$$

(7) A heavy uniform rod is supported in a horizontal position by three equal forces, one acting at one end, the other two

at the other: to shew that the angle between the directions of the two latter forces must be 120°.

Let AB, fig. (76), be the rod, P the force acting at the end B: let O be the intersection of P's direction with that of the weight W of the rod supposed to be collected at its middle point C. Now, in order that the rod may be at rest, the direction of the resultant of the two forces P, P, acting at A, must pass through O, and its magnitude must be equal to P: this cannot be the case unless the two forces at A are inclined to the line OA, on opposite sides of the line, at angles of 60°. The angle between the two forces must therefore be 120°.

(8) A heavy uniform rod is capable of moving in a vertical plane about a hinge at one extremity. A string, fastened to the other extremity, passes over a smooth peg vertically above the hinge and at a distance from it equal to the length of the rod, and has a weight P attached to it: to determine the position of equilibrium of the rod, which is supposed to be inclined to the vertical, and its pressure on the hinge.

Let AB, fig. (77), be the rod, C the peg; let W be the weight of the rod, which we may suppose to be suspended from its middle point G: produce the vertical line through G to intersect BC in D: it is evident that D is the middle point of BC. The tension of the string will exert a force P on the end B of the rod in the direction BC.

Since the directions of P and W pass through D, it is essential to equilibrium that the direction of the reaction R, exerted by the hinge on the rod, also pass through D. Let $\angle BAC = \theta$, and therefore $\angle CAD = \frac{\theta}{2}$.

Since P, W, R, act on the rod in directions parallel to the sides DC, CA, AD, respectively, of the triangle CAD, we have

$$\frac{P}{W} = \frac{DC}{CA} = \sin\frac{\theta}{2},$$

which determines the position of the rod's equilibrium. We have also

$$\frac{R}{W} = \frac{AD}{CA} = \cos\frac{\theta}{2},$$

whence $$R = \left(W^2 - W^2 \sin^2 \frac{\theta}{2}\right)^{\frac{1}{2}}$$
$$= (W^2 - P^2)^{\frac{1}{2}}.$$

The expression for R may be obtained also in the following manner.

Since P and R are at right angles to each other, their resultant must be equal to

$$(P^2 + R^2)^{\frac{1}{2}}:$$

but this resultant must be equal to W: hence

$$W = (P^2 + R^2)^{\frac{1}{2}},$$

and therefore

$$R = (W^2 - P^2)^{\frac{1}{2}}.$$

(9) In a vertical heavy plane a given triangle ABC is cut out, so that the centre of gravity of the mass is at C: the sides AC, BC, of the triangle rest on two pegs P, Q: to find the position of equilibrium of the mass.

The reactions of the pegs P, Q, on the mass act normally to the sides AC, BC, of the triangle: let O, fig. (78), be the point of intersection of the normals at P, Q: join CO, PQ. Since the directions of the two reactions pass through O, the vertical through C, in which the weight of the mass acts, must also pass through O.

Let $\angle ACO = \theta$, $\angle BCO = \phi$, $\angle ACB = \omega$; and let α denote the inclination of PQ to the horizon.

From the geometry we see that

$$\angle PEC = \angle POE + \angle OPE$$
$$= \angle POC + \angle OPQ:$$

but, since a circle may be described about the quadrilateral $POQC$, the angle OPQ is equal to the angle OCQ: hence

$$\angle PEC = \angle POC + \angle OCQ,$$

and therefore

$$\frac{\pi}{2} + \alpha = \frac{\pi}{2} - \theta + \phi,$$

or $$\phi - \theta = \alpha:$$

but $$\phi + \theta = \omega:$$

hence $$\theta = \frac{\omega - \alpha}{2}, \quad \phi = \frac{\omega + \alpha}{2}:$$

these values of θ, ϕ, define the position of equilibrium.

(10) A rod AB, fig. (79), without weight, has a smooth ring at the end A and a given weight at the end B: a string is fastened to the end B, passes over a smooth fixed peg, through the ring at A, and is attached to another given weight: to determine the position of equilibrium.

Let C be the fixed peg, CH a vertical line, cutting AB in H: let P, W, be the given weights. The forces acting on the rod AB are indicated in the diagram.

Since two forces act on the rod at B, the resultant of the two forces P, P, at A, must pass through B, and therefore the directions of the two forces at A must be equally inclined to AB. Hence $\angle CAB = \angle PAB = \angle AHC$: and therefore also $CH = CA$. The resultant of the two forces at A is equal to $2P \cos A$. Thus the directions of the three forces $2P \cos A$, P, W, pass through the point B, and are parallel respectively to the sides HB, BC, CH, of the triangle HBC. Hence

$$\frac{2P \cos A}{W} = \frac{HB}{CH} = \frac{\sin(A - B)}{\sin B} \quad \text{..............} \quad (1),$$

and $$\frac{W}{P} = \frac{CH}{BC} = \frac{CA}{BC} = \frac{\sin B}{\sin A} \quad \text{..............} \quad (2).$$

From (1) we have

$$\frac{2P}{W} = \tan A \cot B - 1,$$

$$\tan A = \tan B \,.\, \frac{W + 2P}{W} \quad \text{..........................} \quad (3),$$

and therefore, from (2),

$$\cos B = \frac{W + 2P}{P} \cos A \quad \text{...........................} \quad (4).$$

Eliminating B between (2) and (4) we get

$$1=\frac{(W+2P)^2}{P^2}\cos^2 A+\frac{W^2}{P^2}\sin^2 A,$$

$$P^2=W^2+4P(P+W)\cos^2 A,$$

$$\cos A=\frac{1}{2}\left(\frac{P-W}{P}\right)^{\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Again, $CH=AC=AB\,.\,\dfrac{\sin B}{\sin(A+B)}=\dfrac{AB}{\cos A\,(1+\tan A\cot B)}$

$$=\frac{W\,.\,P^{\frac{1}{2}}\,.\,AB}{(P+W)(P-W)^{\frac{1}{2}}},$$

by (3) and (5).

Also $$BC=\frac{P}{W}\,.\,AC=\frac{P^{\frac{3}{2}}\,.\,AB}{(P+W)(P-W)^{\frac{1}{2}}}.$$

The expressions for AC, BC, and $\cos A$, determine the lengths of the portions of the string between the peg and the ends of the rod, and also the inclination of the rod to the vertical; the position of equilibrium is therefore completely ascertained.

(11) On the surface of a table are three smooth pegs: a string is passed round them and its ends pulled in opposite directions along the same straight line: to determine the pressure upon each peg and to shew how the pressures combine to neutralize each other.

Let A, B, C, fig. (80), be the three pegs, and let P be the force with which each end of the string is pulled.

Then, the tension of the string being the same throughout, the pressure on the peg A will be the resultant of two equal forces P, P, inclined to each other at an angle A: the pressure on A will therefore be equal to $2P\cos\dfrac{A}{2}$: similarly, the pressures on B, C, will be respectively equal to

$$2P\cos\frac{B}{2},\quad 2P\cos\frac{C}{2}.$$

The resultant of all the forces acting on the string is equivalent to the resultant of the three forces,

$$2P\cos\frac{A}{2},\quad 2P\cos\frac{B}{2},\quad 2P\cos\frac{C}{2},$$

acting respectively along the lines OA, OB, OC, which bisect the angles of the triangle ABC.

Now $\sin\angle BOC = \sin\left(\frac{A}{2}+\frac{B}{2}+\frac{A}{2}+\frac{C}{2}\right) = \sin\left(\frac{\pi}{2}+\frac{A}{2}\right) = \cos\frac{A}{2}$.

Similarly, it may be shewn that $\sin\angle COA$, $\sin\angle AOB$, are equal to $\cos\frac{B}{2}$, $\cos\frac{C}{2}$, respectively.

Hence the forces acting on the string in the directions OA, OB, OC, are proportional to

$$\sin\angle BOC,\quad \sin\angle COA,\quad \sin\angle AOB,$$

and therefore balance one another.

(12) A uniform beam rests with one extremity on a horizontal plane, its other end resting against a vertical wall: assuming the coefficient of friction to be the same at both ends, to determine the limiting position of equilibrium of the beam.

Let AB, fig. (81), be the beam; let AO, BO, be the directions of the reactions of the horizontal plane and vertical wall CX, CY. Join GO, G being the middle point of the beam. Then for equilibrium it is necessary that GO be a vertical line.

The inclination of AO to AC and of BO to BY will be $\cot^{-1}\mu$, μ being the coefficient of friction. Let $\angle BAC = \theta$.

Then, from the triangle AOG, denoting $\cot^{-1}\mu$ by α, we have

$$\frac{OG}{AG} = \frac{\sin(\alpha-\theta)}{\cos\alpha},$$

and, from the triangle BOG,

$$\frac{OG}{BG} = \frac{\cos(\theta-\alpha)}{\sin\alpha}.$$

Hence, AG being equal to BG, we see that

$$\frac{\sin(\alpha-\theta)}{\cos\alpha}=\frac{\cos(\theta-\alpha)}{\sin\alpha},$$

$$\tan(\alpha-\theta)=\cot\alpha,$$

$$\alpha-\theta=\frac{\pi}{2}-\alpha,$$

$$\theta=2\alpha-\tfrac{1}{2}\pi=2\cot^{-1}\mu-\tfrac{1}{2}\pi.$$

(13) A rigid weightless rod is suspended by two strings, which are attached to its ends and to a fixed point; find at what point of the rod a weight must be fastened, (1) so that it may be horizontal, (2) so that the tensions of the strings may be equal.

Let O be the point of suspension, AB the rod: from O draw OP at right angles to the rod, and OQ bisecting the angle AOB: then the points P, Q, in which these two lines intersect the rod, are the two required points.

(14) One end of a string is nailed to a point in the surface of a sphere, the other end being fixed to a point in a smooth vertical wall, against which the sphere presses: find the tension of the string and the pressure of the sphere on the wall.

Let $r=$ the radius of the sphere, $l=$ the length of the string, $W=$ the weight of the sphere, $T=$ the tension of the string, and $R=$ the pressure on the wall: then

$$T=\frac{W(l+r)}{(l^2+2lr)^{\frac{1}{2}}},\quad R=\frac{Wr}{l+r}.$$

(15) A fly stands upon a needle, the ends of which rest on a smooth circular arc, the plane of which is vertical: find the position of equilibrium of the needle.

If c represent the distance of the centre of the circle from the needle, and a, b, the distances of the centre of gravity of the fly and needle from the ends of the needle, the inclination of the needle to the vertical is equal to

$$\tan^{-1}\frac{2c}{a\sim b}.$$

(16) A smooth wire is bent into the form of a circle, and is supported by a small ring, which slides on it and which is attached by a fine string to a vertical wall, against which the circle rests: find the inclination of the string to the wall, when the tension is double the weight of the circle.

The required angle of inclination is $\frac{\pi}{3}$.

(17) OB, fig. (82), is a string sustaining a uniform rod AB against a smooth wall OC: find the position of equilibrium.

If $OB = b$, $AB = 2a$, $\angle AOB = \theta$, $\angle BAC = \phi$; then

$$\sin\theta = \left(\frac{16a^2 - b^2}{3b^2}\right)^{\frac{1}{2}}, \quad \sin\phi = \left(\frac{16a^2 - b^2}{12a^2}\right)^{\frac{1}{2}}.$$

(18) A uniform beam is hung from a fixed point by two unequal strings attached to its extremities: compare the tension of each string with the weight of the beam.

If a, b, represent the lengths of the strings, P, Q, their respective tensions, c the length and W the weight of the beam, then

$$\frac{P}{W} = \frac{a}{(2a^2 + 2b^2 - c^2)^{\frac{1}{2}}}, \quad \frac{Q}{W} = \frac{b}{(2a^2 + 2b^2 - c^2)^{\frac{1}{2}}}.$$

(19) The line of intersection of two smooth planes A, B, is horizontal; a rod CD rests on the two planes, first, with its extremity C in contact with the plane A, and, secondly, with its extremity D in contact with the same plane. If θ, ϕ, be the inclinations of the rod to the horizon in these two positions of equilibrium, prove that $\tan\theta + \tan\phi$ is invariable, whatever be the length of the rod or the position of its centre of gravity.

(20) Prove that no uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl, at an inclination to the horizon greater than $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

(21) A heavy sphere hangs from a peg by a string, the length of which is equal to the radius, and rests against another peg vertically below the former, the distance between the

two being equal to the diameter: find the tension of the string and the pressure on the lower peg.

The tension of the string is equal to the weight and the pressure on the peg to half the weight of the sphere.

(22) One end of a uniform straight rod rests against a smooth vertical wall: a smooth ring without weight, attached to a point in the wall by a fine inextensible string, slides on the rod: if θ be the angle which the rod, when in equilibrium, makes with the wall, and if the length of the string be $\left(\frac{1}{2n}\right)^{\text{th}}$ of that of the rod, prove that

$$\cot^3\theta + \cot\theta = n.$$

(23) A uniform heavy beam is suspended by strings attached to its extremities, passing over a smooth peg, and having unequal weights attached to their other extremities, so that the weights hang down on opposite sides of the peg: find the sides of the triangle formed by the beam and the strings, when the system is in equilibrium.

If P, Q, be the two unequal weights, W the weight of the beam and c its length, the two other sides of the triangle are equal to

$$\frac{Pc}{(2P^2 + 2Q^2 - W^2)^{\frac{1}{2}}}, \qquad \frac{Qc}{(2P^2 + 2Q^2 - W^2)^{\frac{1}{2}}}.$$

(24) A cone, the vertical angle of which is $\cos^{-1}\frac{1}{3}$, is enclosed in the circumscribing spherical surface, which is fixed; shew that it will rest in any position.

(25) A square rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a string the length of which is equal to a side of the square: shew that the distances of three of its angular points from the wall are as 1, 3, and 4.

(26) A uniform isosceles triangle rests with its base horizontal on one inclined plane and its vertex on another: if α be the inclination of the latter plane and $\frac{1}{2}\pi - \alpha$ of the former,

determine the inclination of the plane of the triangle to the horizon.

If θ be the inclination of the plane of the triangle to the horizon,

$$\tan\theta = \tfrac{1}{3}(\tan\alpha - 2\cot\alpha).$$

(27) A heavy equilateral triangle, hung up on a smooth peg by a string the ends of which are attached to two of its angular points, rests with one of its sides vertical; shew that the length of the string is double the altitude of the triangle.

(28) A lamina, cut into the form of an equilateral triangle, is hung up against a smooth vertical wall by means of a string attached to the middle point of one side, so as to have a corner in contact with the wall; shew that, when there is equilibrium, the reaction of the wall and the tension of the string are independent of the length of the string, and that, if the string exceed the length of a side of the triangle, equilibrium in such a position is impossible.

CHAPTER VIII.

EQUILIBRIUM OF A SYSTEM OF BODIES.

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(1) An inextensible string binds tightly together two smooth cylinders, the ratio between radii of which is given: to find the ratio of the mutual pressure of the cylinders to the tension of the string.

Let A, B, fig. (83), be the centres of transverse circular sections of the cylinders, in the plane of the string: produce the common tangent HK to meet AB, produced, in C: from A, B, draw AH, BK, to the points of contact H, K: draw KL, parallel to BA, to meet AH in L. Let R denote the action of the cylinder A upon the cylinder B, T being the tension exerted upon the latter cylinder by each of the two rectilinear portions of the string. Let r, s, be the radii of A, B, respectively, and let $\angle ACH = \theta$. Then, for the equilibrium of the cylinder B, we have, resolving along AC,

$$R = 2T \cos \theta.$$

But $$\sin \theta = \frac{HL}{KL} = \frac{r-s}{r+s},$$

and therefore

$$\cos \theta = \frac{2(rs)^{\frac{1}{2}}}{r+s} = \frac{2}{\left(\frac{r}{s}\right)^{\frac{1}{2}} + \left(\frac{s}{r}\right)^{\frac{1}{2}}}:$$

hence $$\frac{R}{T} = \frac{4}{\left(\frac{r}{s}\right)^{\frac{1}{2}} + \left(\frac{s}{r}\right)^{\frac{1}{2}}}.$$

(2) Three equal heavy cylinders, each of which touches the other two, are bound together by a string and laid upon a horizontal plane; the tension of the string being given, to find the pressures between the cylinders.

Let W be the weight of each cylinder, T the tension of the string, R the reaction between the higher and each of the lower cylinders, S the reaction between the two lower, R' the reaction of the horizontal plane upon each of the lower cylinders. The forces acting upon each cylinder are exhibited in diagram (84).

The two forces T, T, which act upon the higher cylinder, may be replaced by a vertical force $2T\cos 30°$, acting downwards in the line of W's action: hence, for the equilibrium of the higher cylinder, we have, resolving forces vertically,

$$2R\cos 30° = 2T\cos 30° + W,$$

or

$$R = T + \frac{W}{\sqrt{3}} \text{ .... } (1).$$

Again, the two forces T, T, which act upon either of the lower cylinders, may be replaced by a force $2T\cos 30°$, the direction of which bisects the angle between R and S: hence, resolving horizontally the forces which act on one of the lower cylinders, we have

$$S + R\cos 60° = 2T\cos 30° . \cos 30°,$$

or

$$S + \tfrac{1}{2}R = \tfrac{3}{2}T \text{ .... } (2).$$

From (1) and (2), we have

$$S = T - \frac{W}{2\sqrt{3}} \text{ .... } (3).$$

The equations (1) and (3) give the pressures between the cylinders.

COR. Resolving vertically the forces which act on one of the lower cylinders, we have

$$R' + 2T\cos 30° . \sin 30° = W + R\sin 60°,$$

$$R' + \frac{\sqrt{3}}{2} . T = W + \frac{\sqrt{3}}{2} . R,$$

and therefore, by (1),

$$R' = \tfrac{3}{2}W.$$

Thus the sum of the pressures on the horizontal plane is equal to $3W$, the weight of the three cylinders; as might have been anticipated.

(3) A uniform beam is moveable round a hinge, which is fixed on a smooth inclined plane: the other end presses against a right-angled cone, of equal weight, resting on the plane, with which its base is in contact: the inclination of the beam to the plane being the same as that of the plane to the horizon, to find the inclination of the plane to the horizon.

Let R, fig. (85), be the mutual action between the beam and the cone: let W be the weight of the cone or beam: and θ the inclination of the plane to the horizon.

Then, for the equilibrium of the beam, taking moments about the hinge, we have, $2a$ being the length of the beam,

$$R \cdot 2a \cos\left(\frac{\pi}{4} - \theta\right) = Wa \cos 2\theta,$$

and, for the equilibrium of the cone, resolving the forces, which act on it, parallel to the plane,

$$W \sin\theta = R \sin\frac{\pi}{4}.$$

Hence
$$2\sin\theta\cos\left(\frac{\pi}{4} - \theta\right) = \sin\frac{\pi}{4}\cos 2\theta,$$

$$2\sin\theta(\cos\theta + \sin\theta) = \cos^2\theta - \sin^2\theta,$$

$$2\sin\theta = \cos\theta - \sin\theta, \quad \tan\theta = \tfrac{1}{3},$$

and therefore the plane's inclination to the horizon is equal to $\tan^{-1}(\frac{1}{3})$.

(4) Two equal uniform rods AA', BB', fig. (86), are attached to smooth hinges at A, B, in a horizontal line, their lower ends being tied together by a fine string: a sphere is placed upon the two rods: to find the tension of the string.

Let R be the mutual action between each rod and the sphere at the point of contact E. Let W be the weight of the sphere and P of each rod. Let T be the tension of the string.

Let $AA' = 2a = BB'$, and $\alpha =$ the inclination of each rod to the vertical. Let $2c$ denote the distance between A and B, and $2c' =$ the length of the string, and let $r =$ the radius of the sphere.

For the equilibrium of the sphere, we have, resolving vertically,

$$2R \, . \sin \alpha = W,$$

and, for the equilibrium of either rod, taking moments about its higher end,

$$T \, . \, 2a \cos \alpha = R \, . \, AE + P \, . \, a \sin \alpha.$$

From these two equations we see that

$$T = \frac{W}{4a \sin \alpha \cos \alpha} \, . \, AE + \tfrac{1}{2} P \, . \tan \alpha :$$

but, from the geometry,

$$AE \, . \sin \alpha + r \cos \alpha = c;$$

hence
$$T = W \, . \frac{c - r \cos \alpha}{4a \sin^2 \alpha \, . \cos \alpha} + \tfrac{1}{2} P \, . \tan \alpha.$$

The value of α is known from the equation

$$\sin \alpha = \frac{c - c'}{2a}.$$

(5) AC, BC, fig. (87), two equal and uniform beams, connected by a hinge at C, are placed on the circumference of a circle, the plane of which is vertical, the point C being in the vertical line through the centre of the circle: to find the position of equilibrium.

Let W, the weight of each beam, be supposed to be collected at its centre of gravity; let R be the reaction of the circle on each beam. The action of the beam BC upon the beam AC must pass through the intersection E of the directions of the forces R and W, which act on AC. Similarly, the action of AC upon BC must pass through the intersection F of the directions of the forces R and W, which act on BC. But the points E

and F, as appears from the geometry, lie in a horizontal line: hence, the action and reaction of the two beams against each other being necessarily opposite forces, it follows that the point C must be in the same horizontal line with the points E and F.

Let $\angle COE = \theta$, $r =$ the radius of the circle, $CG = a$: then, from the geometry, we see that

$$\cos\theta = \frac{CE}{CG} = \frac{CO\tan\theta}{CG} = \frac{r\tan\theta}{a\cos\theta},$$

$$a\cos^3\theta = r\sin\theta,$$

an equation which determines the position of equilibrium.

COR. Let S denote the action of the beam CA on the beam CB: then we have

$$R : S : W :: 1 : \sin\theta : \cos\theta,$$

whence $$R = W\sec\theta,$$

and $$S = W\tan\theta.$$

(6) Two equal circular disks with smooth edges, placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle, touch each other in the line bisecting the angle: to find the radius of the least disk which may be pressed between them without causing them to separate.

Let R, fig. (88), denote the action of each vertical plane upon the disk which is in contact with it, S the mutual action between the two disks, T the action of the third disk upon each of the other two. Let r denote the radius of each of the two former disks, O, O, and r' that of the third disk C.

$$\text{Let } \angle BAC = \alpha, \ \angle ACO = \theta.$$

Then, for the equilibrium of each of the disks O, O, we have, resolving perpendicularly to the line of T's action,

$$S\cos\theta = R\cos(\alpha + \theta).$$

Now it is necessary for equilibrium that S, as given by this formula, be positive: hence

$$\alpha + \theta < \frac{\pi}{2},$$

whence $$\sin\theta < \cos\alpha,$$

and therefore $$\frac{r}{r+r'} < \cos\alpha,$$

$$r(1-\cos\alpha) < r'\cos\alpha,$$

$$r' > r(\sec\alpha - 1).$$

Thus we see that the radius of the third disk must not be shorter than $r(\sec\alpha - 1)$.

(7) A string of equal spherical beads is placed upon a smooth cone, of which the axis is vertical, the beads being just in contact with each other, so that there is no mutual pressure between them: to find the tension of the string, and to deduce the limiting value, when the number of beads is indefinitely great.

Let n denote the number of the sides of the horizontal polygon, of which the centres of the beads are the angular points: let W be the weight of each bead, β half the vertical angle of the cone, R its reaction on each of the beads, T the tension of the string. Then, 2α denoting the angle between two successive sides of the polygon, we have, resolving horizontally the forces which act on any bead,

$$2T\cos\alpha = R\cos\beta,$$

and, resolving vertically,

$$W = R\sin\beta:$$

whence $$T = \frac{W\cot\beta}{2\cos\alpha}:$$

but $$\alpha = \frac{\pi}{2} - \frac{\pi}{n};$$

hence $$T = \frac{W\cot\beta}{2\sin\frac{\pi}{n}}.$$

When the number of beads is indefinitely great, P denoting the weight of all the beads together,

$$T = \frac{\pi}{n} \cdot \frac{n\,W \cot\beta}{2\pi \sin\frac{\pi}{n}} = \frac{P\cot\beta}{2\pi}.$$

(8) A cylindrical shell without bottom stands on a horizontal plane, and two smooth spheres are placed within it, of which the diameters are each less while their sum is greater than that of the interior surface of the shell; to shew that the cylinder will not upset, if the ratio of its weight to the weight of the upper sphere be greater than $2c-a-b$ to c, where a, b, c, are the radii of the spheres and cylinder.

Let R, fig (89), be the mutual pressure between the cylinder and the upper sphere and R' between the cylinder and the lower sphere; let S be the mutual pressure between the two spheres: let a be the radius of the lower and b of the upper sphere, W the weight of the cylinder, α the inclination of the distance between the centres of the spheres to the horizon.

For the equilibrium of the upper sphere, we have, resolving horizontally,

$$R = S\cos\alpha \quad \text{.............................} \quad (1),$$

and, resolving vertically,

$$P = S\sin\alpha \quad \text{.............................} \quad (2);$$

whence

$$R = P\cot\alpha \quad \text{.............................} \quad (3).$$

For the equilibrium of the lower sphere, resolving horizontally,

$$R' = S\cos\alpha \quad \text{.............................} \quad (4).$$

From (1) and (4) we see that $R' = R$: and therefore, the line of action of R being higher than that of R', R and R' tend to upset the cylinder about the point C of the base vertically below its contact with the upper sphere: and, when W is the least possible consistently with the equilibrium of the cylinder, the horizontal plane will exert force on the cylinder only at C. Under these circumstances we have, taking the moments of the forces on the cylinder about C,

$$R\{(a+b)\sin\alpha + a\} = Ra + Wc,$$

and therefore, by (3),

$$\cos\alpha\{(a+b)\sin\alpha+a\}=a\cos\alpha+\frac{W}{P}c\sin\alpha,$$

$$(a+b)\cos\alpha=\frac{Wc}{P}:$$

but, by the geometry,

$$a+b+(a+b)\cos\alpha=2c;$$

hence
$$\frac{Wc}{P}=2c-a-b,$$

$$\frac{W}{P}=\frac{2c-a-b}{c}.$$

(9) Three rods, jointed together at their extremities, are laid on a smooth horizontal table; and forces are applied at the middle points of the sides of the triangle formed by the rods, respectively perpendicular to them: to shew that, if these forces produce equilibrium, the strains at the joints will be equal to one another, and that their directions will touch the circle circumscribing the triangle.

Let ABC, fig. (90), be the triangle of rods. Let the strain on the end A of the rod AB meet the perpendicular to AB, drawn through its middle point, in the point C': then, for the equilibrium of AB, it is necessary that the strain on the end B of the rod AB shall act in the direction BC'.

Since the strains in AC', BC', are counteracted by a force through C' bisecting the angle C', these strains must be equal to each other: in like manner we may shew that the strain at B is equal to that at C, and the strain at C to that at A. Let S denote each of these equal strains; P, Q, R, the normal forces on BC, CA, AB, respectively. Let λ, μ, ν, be the angles at the respective bases of the isosceles triangles $BA'C$, $CB'A$, $AC'B$; and α, β, γ, the angles of the triangle ABC.

For the equilibrium of CA, we have, resolving at right angles to it,

$$2S\sin\mu=Q,$$

and similarly, for the equilibrium of AB,

$$2S\sin\nu=R:$$

hence $$\sin\mu : \sin\nu :: Q : R:$$

but, since P, Q, R, which are at right angles to the sides of the triangular system of rods, produce equilibrium,

$$Q : R :: CA : AB :: \sin\beta : \sin\gamma:$$

hence $$\sin\mu : \sin\nu :: \sin\beta : \sin\gamma,$$

$$\sin\mu \sin\gamma = \sin\nu \sin\beta,$$

$$\cos(\mu-\gamma) - \cos(\mu+\gamma) = \cos(\nu-\beta) - \cos(\nu+\beta):$$

but, from the geometry, we see that

$$\mu+\nu=\beta+\gamma, \quad \mu-\gamma=\beta-\nu:$$

hence $$\cos(\mu+\gamma) = \cos(\nu+\beta),$$

$$\mu+\gamma=\nu+\beta:$$

but $$\mu-\gamma=\beta-\nu;$$

hence $$\mu=\beta, \ \nu=\gamma;$$

similarly we may shew that $\lambda = \alpha$.

These results shew that $B'C'$, $C'A'$, $A'B'$, the directions of the strains at A, B, C, are tangents to the circle which circumscribes the triangle ABC.

(10) Two weights P and Q are suspended by strings tied round the circumferences of two rough wheels, which act upon an intermediate rough wheel: the wheels are moveable about smooth pivots at their centres, the planes of the wheels being vertical: to find the relation between P and Q, when there is equilibrium.

Let A, B, fig. (91), be the centres of the wheels from which P, Q, are suspended, and C the centre of the intermediate wheel. Let a, b, c, be the radii of the three wheels A, B, C, respectively.

Let S denote the mutual tangential action between the wheels A, C, and T that between the wheels B, C.

Taking moments, about the centre of each of the wheels, of the forces acting on it, we have

$$Pa = Sa, \quad Sb = Tb, \quad Tc = Qc:$$

hence $P = S, \quad S = T, \quad T = Q,$

and therefore $P = Q.$

(11) Two uniform heavy straight rods, of equal length, are placed on a rough horizontal plane, their upper extremities resting against each other: to determine the least inclination to the horizon at which they can rest.

The reactions of the horizontal plane on the ends A, A', fig. (92), of the rods are both inclined to the horizon, when the angles CAA', $CA'A$, are the least possible consistently with equilibrium, at an angle $\tan^{-1}\left(\frac{1}{\mu}\right)$, μ being the coefficient of friction. Let the reactions meet the verticals through G, G', the centres of gravity of the rods, in O, O'. Then, as the condition of equilibrium, the direction of the mutual actions of the two rods at C must pass through O, O'; but O, O', are in a horizontal line: hence OCO' is a horizontal straight line. Draw GH horizontally to meet AO in H. Then, from the triangle OHG, we have

$$\frac{1}{\mu} = \tan \angle OHG = \frac{OG}{GH} = \frac{OG}{\frac{1}{2} OC} = 2 \tan \phi,$$

where ϕ is the required inclination of each rod to the horizon; hence

$$\phi = \tan^{-1}\left(\frac{1}{2\mu}\right).$$

(12) Two rough bodies rest on an inclined plane, and are connected by a string which is parallel to the plane: if the coefficient of friction be not the same for both, to find the greatest inclination of the plane which is consistent with equilibrium.

Suppose the rougher of the two bodies to be at a higher point of the plane than the other body: this supposition will enable the less rough of the two bodies to escape sliding for a greater elevation than if it were higher up the plane than the rougher body.

Let W, W', be the weights of the rougher and less rough bodies respectively, R, R', the respective reactions of the plane upon them, μ, μ', their coefficients of friction, T the tension of

the string, α the greatest inclination of the plane which is consistent with equilibrium.

For the equilibrium of the rougher body we have, resolving parallel and at right angles to the plane,

$$\mu R = T + W \sin \alpha,$$
$$R = W \cos \alpha,$$

and therefore

$$\mu W \cos \alpha = T + W \sin \alpha \quad \text{.....................} \quad (1).$$

Similarly, for the less rough body, putting $-T$ for T,

$$\mu' W' \cos \alpha = -T + W' \sin \alpha \quad \text{..................} \quad (2).$$

Adding together the equations (1) and (2), we get

$$(\mu W + \mu' W') \cos \alpha = (W + W') \sin \alpha,$$

and therefore $$\tan \alpha = \frac{\mu W + \mu' W'}{W + W}.$$

(13) If an inextensible string be wound round two spheres in contact, and be tightened so as to have a given tension, prove that the mutual pressure of the spheres is greatest when the radii are equal.

(14) OA, OB, are respectively the vertical and horizontal radii of a quadrant AB of a circle; P and Q are two weights, connected by a string; P hangs freely along AO, Q rests on the convex side of the quadrant: prove that there will be equilibrium if

$$\angle AOQ = \sin^{-1}\left(\frac{P}{Q}\right).$$

(15) Two smooth cylinders, of equal radii, just fit in between two parallel vertical walls, and rest on a horizontal plane without pressing against the walls: if a third such cylinder be placed on the top of them, find the resultant pressure against either wall.

If W = the weight of each cylinder, the pressure on either wall is equal to $\frac{W}{2\sqrt{3}}$.

(16) Two inclined planes, of given equal bases and given equal altitudes, are placed facing each other on a smooth table: a sphere of given weight is placed between them, not touching the table, and they are prevented from sliding further apart by a string, which ties them together: find the tension of the string.

If h be the height of each plane, and b the length of its base, then, W being the weight of the sphere, the tension of the string is equal to

$$\frac{1}{2} W \frac{h}{b}.$$

(17) An isosceles triangle, without weight, stands with its base on a horizontal plane; and two equal heavy uniform beams, in the same vertical plane with it, rest against its sides and upon the plane at equal elevations, and are prevented from slipping by obstacles at their lower ends: find the pressure of the triangle upon the plane.

If $\alpha =$ the inclination of either equal side of the triangle, and β of either beam to the horizon, then, W denoting the weight of either beam, the required pressure is equal to

$$\frac{W}{1 + \tan\alpha \tan\beta}.$$

(18) A thread, passing over a hoop, fig. (93), of which the plane is vertical, is held to the hoop by two equal rings P_1, P_2, and a third ring P_3, equal to each of the others, hangs on the thread between the two; prove that, if Q be the point in which a tangent to the hoop, parallel to P_1P_2, meets the vertical through P_3, then P_3 is situated at the centre of gravity of the triangle P_1QP_2.

(19) The extremities of a string without weight are fastened to two equal heavy rings, which slide on smooth fixed rods in the same vertical plane, equally inclined to the vertical; and to the middle point of the string a weight is fastened, equal to twice the weight of either ring; find the position of equilibrium and the tension of the string.

If α be the inclination of each rod, and θ of each portion of the string to the vertical, and T the tension of the string,

$$\tan\theta = 2\cot\alpha, \text{ and } T = W(1 + 4\cot^2\alpha)^{\frac{1}{2}}.$$

(20) If, in the preceding problem, the point to which the weight is fastened be not the middle point of the string, shew that, in the position of equilibrium, the tensions of its two portions will still be equal.

(21) Two small rings rest symmetrically on the circumference of a smooth vertical circle, being attached to the highest point by means of two strings, each equal to the radius of the circle: each of the strings passes through another ring, the weight of which is double that of either of the others, and which hangs freely: prove that, when there is equilibrium, the three rings lie in the same horizontal line.

(22) Two weights P and W, connected by a string, rest in equilibrium on a smooth fixed sphere, and the radii, at the points where they rest, make angles α, β, with the horizon; but, if the positions of the weights be changed, W will just be supported by a weight Q: prove that

$$P\cos^2\alpha = Q\cos^2\beta.$$

If the sphere be rough, the coefficient of friction $\tan\lambda$, and if the greatest weight which P can support be equal to the least with which Q can be in equilibrium, when they occupy the same positions as before, prove that

$$P\cos^2(\alpha-\lambda) = Q\cos^2(\beta+\lambda).$$

(23) Three straight tobacco-pipes rest upon a table, with their bowls, mouth downwards, in the angles of an equilateral triangle, the tubes being supported in the air by crossing symmetrically, each under one and over the other, so as to form another equilateral triangle: prove that the mutual pressure of the tubes varies inversely as the side of the last triangle.

(24) A sphere rests upon a fixed horizontal plane: two equal rods, connected together, at their higher ends by a smooth hinge, rest symmetrically across the sphere, their lower ends touching without pressing the horizontal plane: find the inclination of either rod to the vertical.

If θ denote the inclination of either rod to the vertical,

$$\sin\theta = \sqrt{3} - 1.$$

(25) A rod OA, fig. (94), moveable about a hinge at O, leans against a circular board, which rests on a smooth horizontal plane OC and against a vertical plane CF: find the pressures of the board on the horizontal and vertical planes.

Let $OA = 2a$, $\angle AOC = \alpha$, $r =$ the radius and $W =$ the weight of the board, $W' =$ the weight of OA: then the pressures on the horizontal and vertical planes are respectively equal to

$$W + W' . \frac{a}{r} . \cos^2\alpha . \tan\frac{\alpha}{2},$$

and
$$2W'\frac{a}{r}\sin^2\frac{\alpha}{2}\cos\alpha.$$

(26) A given uniform rod OA, fig. (95), moveable about a hinge at O, presses upon a given sphere C, which is at rest on an inclined plane OB: supposing the angle AOB to be known, find the inclination of the plane to the horizon.

Let P denote the weight and $2a$ the length of the rod; W the weight and r the radius of the sphere, α the angle AOB, and θ the inclination of OB to the horizon: then

$$\tan\theta = \frac{2\sin^2\frac{\alpha}{2} . \cos\alpha}{\frac{Wr}{Pa} + 2\sin^2\frac{\alpha}{2} . \sin\alpha}.$$

(27) Three equal spheres lie glued together, in contact, on a horizontal plane: a fourth equal sphere lies upon them: find the force exerted by the glue between each pair of the spheres, which lie on the horizontal plane, to prevent their separation.

The required force is equal to $\frac{W}{3\sqrt{6}}$, W being the weight of each of the spheres.

(28) A smooth globe, of which the centre is C, fig. (96), and the weight of which is P, rests symmetrically upon the ends of two equal smooth rods AB, $A'B'$, moveable about hinges at A, A', which are in a horizontal line, the weight of each rod being W. If θ, ϕ, denote the inclinations of BC, BA, respectively, to the horizon, prove that

$$\frac{\tan\phi}{\tan\theta} = \frac{P + W}{P}.$$

(29) Four smooth equal spheres are at rest in a hemispherical bowl: the centres of three of them are in the same horizontal plane, and the remaining one rests symmetrically upon the others: if the radius of each sphere be one-third of that of the bowl, prove that the mutual pressures of the spheres are all equal.

(30) AB is a smooth vertical rod: equal and similar rods are moveable about their extremities, which are fixed at A, like the ribs of an umbrella: to each of them is attached, at a point of which the distance from A is the same for all, a rod without weight, the other extremity of which is supported by a small ring moveable on AB: prove that the force necessary to raise the ring is independent of its position.

(31) Two balls, of equal weight W, are connected by a string $ACDB$, which passes over two tacks C, D, in a horizontal line; a given sphere, of weight W, being placed on the string at the middle point of CD, find how far A and B must rise before the system comes into a position of equilibrium.

If $CD = 2a$, and $r =$ the radius of the sphere, the required elevation of A and B is equal to

$$\frac{a}{\sqrt{3}} \cdot (2 - \sqrt{3}) + \tfrac{1}{6} r \, (\pi - 2\sqrt{3}).$$

(32) Two spheres are supported by strings attached to a given point, and rest against one another: find the tensions of the strings.

Let P, Q be the weights of the spheres, a, b the respective distances of their centres from the given point, and ω the angle between the strings: then the tensions of the two strings will be respectively equal to

$$\frac{P(P+Q)\,a}{(P^2a^2 + 2PQab\cos\omega + Q^2b^2)^{\frac{1}{2}}},$$

and
$$\frac{Q\,(Q+P)\,b}{(Q^2b^2 + 2\,QPba\cos\omega + P^2a^2)^{\frac{1}{2}}}.$$

(33) Three horizontal levers AEB, BFC, CGD, without weight, the fulcrums of which are E, F, G, act upon one another perpendicularly at B and C respectively, and are kept in equilibrium by weights W, $2W$, at A, D, respectively: if AE, EB, BC, CG, GD, are equal to 1, 2, 7, 2, 3, feet respectively, determine the position of F and the magnitude of the pressure on it.

FC is equal to one foot, and the pressure on F is equal to $\frac{7}{2}W$.

(34) A straight lever is moveable about a fixed point in its length, this point coinciding with the common vertex of two inclined planes: a heavy particle is laid on each plane, and attached by a fine string to an extremity of the arm of the lever, the length of each string being equal to that of the arm to which it is attached: determine the inclination of the lever to the horizon when it is in a position of equilibrium.

If a and b be the lengths of the arms, P and Q the weights of the particles, α and β the inclinations of the planes to the horizon, and θ the inclination of the lever to the horizon,

$$\tan\theta = 2 \cdot \frac{Pa\sin^2\alpha - Qb\sin^2\beta}{Pa\sin 2\alpha + Qb\sin 2\beta}.$$

(35) Three rods, without weight, are connected together at their extremities by free hinges, and, at the middle points of the sides of the triangle thus formed, forces act, all inwards or all outwards, perpendicularly to the sides and proportional to them in magnitude: prove that, if the action of one of the hinges be perpendicular to a side with which it is connected, the triangle is right-angled.

(36) Three equal rods, AB, BC, CD, connected by two free joints at B, C, are attached by smooth hinges to two points A, D, in the same horizontal line: if the rod next to one of the hinges makes an angle α with the horizon and the reaction on the joint at its lower end an angle θ, prove that

$$\tan\theta = \tfrac{1}{2}\tan\alpha.$$

(37) Two heavy bars AC, BC, fig. (97), of uniform material and of the same thickness, are attached by two small hooks to two small fixed rings, A and B, in the same horizontal line, and their other extremities are also hooked together at C; find the magnitude and direction of the force exerted at C; and prove that its direction will fall without or within the triangle ABC, accordingly as the perpendicular from C on AB falls within or without the triangle.

Let XY be the line of action of the force at C: let

$\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle ACB = \gamma$, $\angle ACX = \theta$, $\angle BCY = \phi$;

let P, Q, represent the respective weights of the bars AC, BC, and T the magnitude of the required force. Then the magnitude and direction of the required force are given by the equations

$$\cot\theta = \frac{\tan\alpha - \tan\beta\cos\gamma}{\tan\beta\sin\gamma}, \quad \cot\phi = \frac{\tan\beta - \tan\alpha\cos\gamma}{\tan\alpha\sin\gamma},$$

$$\tfrac{1}{2}P\frac{\cos\alpha}{\sin\theta} = T = \tfrac{1}{2}Q\frac{\cos\beta}{\sin\phi}.$$

(38) Two weights of similar material, connected by a fine smooth string, rest on a rough vertical circular circumference, with which the string is in contact: prove that, if μ be the coefficient of friction, the angle subtended at the centre by the distance between the limiting positions of either weight, is equal to $2\tan^{-1}\mu$.

(39) A horizontal rod is moveable freely round its middle point, which is fixed at the vertex of an inclined plane: a weight P hangs freely from the extremity of the rod which is not directly above the inclined plane, and to the other extremity one end of a string, the length of which is equal to half that of the rod, is fastened: to the free end of this string is attached a weight W, resting on the plane: prove that, if α be the inclination of the plane to the horizon,

$$P = 2W\sin^2\alpha.$$

If the plane were rough, and the coefficient of friction such

that W would just rest on it without any support, prove that the equilibrium would not be disturbed if P were increased by a weight bearing to P the ratio of $\cos 2\alpha : 1$.

(40) A series of n equal balls, of different substances, connected together by strings, are placed on a rough plane inclined to the horizon at an angle α, so as to form a line perpendicular to the intersection of the plane with the horizon: prove that, if μ_r be the coefficient of friction between the r^{th} ball and the plane, and the system be on the point of slipping,

$$n \tan \alpha = \mu_1 + \mu_2 + \mu_3 + \ldots\ldots + \mu_n.$$

CHAPTER IX.

MECHANICAL POWERS.

SECT. 1. *Wheel and Axle.*

(1) A MOUSE, of weight W, clings to the lower circumference of the wheel, in a wheel and axle, and so just supports a weight $5W$, the ratio of the radii of the wheel and axle being 10 to 1; to find the inclination, to the vertical, of the radius of the wheel which passes through the position of the mouse; and to shew that the mouse is in a position of stable equilibrium, but that, if it were on the upper surface of the wheel, at a point vertically above its present position, its equilibrium would be unstable.

Let O, fig. (98), be the common axis of the wheel and axle, OA being the horizontal radius of the axle from the end A of which the weight hangs, and OB the radius of the axle at the end B of which the mouse clings. Let θ be the inclination of OB to the vertical. Taking moments about O, for the equilibrium of the system, we have

$$W \,.\, OB \,.\, \sin\theta = 5W \,.\, OA,$$

and therefore, since $OB = 10\,OA$,

$$\sin\theta = \tfrac{1}{2}, \quad \theta = \frac{\pi}{6}.$$

Suppose the mouse to be placed a little above its present position: then, the moment of W about O being increased, the mouse will descend towards its position of rest, raising $5W$. Again, suppose the mouse to be placed a little below its present position: then, the moment of W about O being diminished, the weight $5W$ will descend, raising the mouse towards its position of rest. Thus we see that the position of equilibrium is stable.

If the mouse cling to the wheel at a point vertically above B, it will be in a position of equilibrium; but, as may readily be seen by reasoning like the above, it will recede further from its position of rest, if slightly displaced either way. Its latter position of equilibrium will therefore be unstable.

(2) What weight, suspended from the axle, can be supported by $1\frac{1}{2}$ lbs., suspended from the wheel, if the radius of the axle is $1\frac{1}{2}$ ft., and the radius of the wheel is $3\frac{1}{4}$ feet?

The required weight = 3 lbs. 4 oz.

(3) Two men, who can exercise forces of 200 lbs. and 248 lbs. each, work at an axle to which two wheels are attached, of 5 feet and 4 feet diameter respectively, the diameter of the axle being 20 inches: find the greatest weight the men can raise by it.

The required weight is $1195\frac{1}{5}$ lbs.

(4) If the difference between the radii of a wheel and axle be eight inches, and the power and weight be as 6 to 7, find the radii.

The radii of the wheel and axle are respectively 4 ft. 8 in. and 4 ft.

(5) If a weight W be kept from sliding down an inclined plane, of inclination α, by a string, which is parallel to the plane, and which passes round a wheel of radius r, find the weight which must hang from an axle of radius r', having a common axis with the wheel, that there may be equilibrium.

The required weight is equal to $\dfrac{r}{r'} \cdot W \sin \alpha$.

(6) If there be a system of wheels and axles, such that the rope, which is wound round the axle of the first, passes over the wheel of the second, that round the axle of the second over the wheel of the third, and so on; prove that the power applied at the first wheel is to the weight supported at the last axle as the product of the radii of the axles to the product of the radii of the wheels.

SECT. 2. *Toothed Wheels.*

Two toothed wheels work against each other: shew that, if the number of the teeth in one be prime to that in the other, before two teeth, which have been in contact once, come into contact again, every tooth of the one wheel will have been in contact with every tooth of the other.

SECT. 3. *Single Moveable Pully.*

(1) A rope passes over a pully; one end is attached to a man, who grasps the other end with both hands; to find the proportion of his weight sustained by each arm, when he exerts the same stress on both.

Let W denote the weight of the man, and let P be the force exerted with each hand: then the tension of the string will be $2P$: but the whole weight of the man is sustained by the sum of the tensions of the two portions of the string: hence

$$2P+2P=W, \quad P=\tfrac{1}{4}W;$$

hence a quarter of the man's weight is sustained by each arm.

(2) A weight W, fig. (99), is suspended from a single moveable pully, which is supported by a weight P hanging over a fixed pully, the strings being parallel: prove that, in whatever position they hang, the position of their centre of gravity is the same.

(3) An endless string hangs at rest, over two pegs in the same horizontal plane, with a heavy pully in each festoon of the string: if the weight of one pully be double that of the other, prove that the angle between the portions of the upper festoon must be greater than 120°.

SECT. 4. *First System of Pullies.*

(1) In a system of three moveable pullies, where each pully hangs by a separate string, a weight of W pounds is suspended

at the first pully and one of $3W$ pounds at the second: to find the power.

Let T_1, T_2, T_3, be the tensions of the strings under the pullies C_1, C_2, C_3, fig. (100), respectively: then

$$T_1 = \tfrac{1}{2}W,$$
$$T_2 = \tfrac{1}{2}(3W + T_1) = \tfrac{1}{2}(3W + \tfrac{1}{2}W) = \tfrac{7}{4}W,$$
$$T_3 = \tfrac{1}{2}T_2 = \tfrac{7}{8}W.$$

But $P = T_3$: hence the power P is equal to $\frac{7}{8}W$.

(2) To find the ratio of the power to the weight of each pully in the system of pullies, in which each pully hangs by a separate string, the pullies being of equal weight, (1) when there is no mechanical advantage, (2) when the power just supports the pullies.

Let P represent the power, W the weight, A the weight of each pully, and n the number of pullies. Then, as is proved in systematic treatises on Statics,

$$2^n(P - A) = W - A.$$

(1) When there is no mechanical advantage, W is not greater than P: but our equation shews that it cannot be less than P: hence $W = P$, and therefore

$$2^n(P - A) = P - A,$$

and therefore $P = A$, or the ratio of the power to the weight of each pully is a ratio of equality.

(2) When the power just supports the pullies, $W = 0$; and therefore

$$\frac{P}{A} = \frac{2^n - 1}{2^n}.$$

(3) If a weight of 160 pounds be supported on an inclined plane by means of a string passing under a fixed pully at the top, and carried vertically upwards to the lowest pully of the system where each pully hangs by a separate string; to find the

number of pullies, when 6 pounds will maintain equilibrium. The height of the plane is 2 yards, and its length 10 feet.

The tension of the string is equal to the component of the weight taken along the inclined plane, that is, to

$$160 \times \frac{6}{10} = 96 \text{ pounds}.$$

Then, putting $P=6$ and $W=96$ in the formula $2^n P = W$, where n denotes the number of the moveable pullies, we have

$$2^n \times 6 = 96, \quad 2^n = 16, \quad n = 4.$$

(4) To determine the relation between the radii of the pullies of the system, in which each pully hangs by a separate string, and the strings are parallel, in order that, if their centres be at any time in a straight line, they may always continue to be so.

Let $c, c_1, c_2, c_3, \ldots$ be the depths of the centres of the pullies, beginning with the highest, below a horizontal plane, and let $r, r_1, r_2, r_3, \ldots$ be the respective radii of the pullies. Then, supposing the centres to be in a straight line, we must have

$$\frac{c_1 - c}{r_1} = \frac{c_n - c_{n-1}}{r_n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where n denotes any one of the natural numbers.

Suppose c to be augmented by a length a; then $c_1, c_2, c_3, \ldots$ will be respectively augmented by lengths $\frac{a}{2}, \frac{a}{2^2}, \frac{a}{2^3}, \ldots\ldots$ Hence, supposing the relation (1) still to hold good, we must have

$$\frac{\frac{1}{2}a - a}{r_1} = \frac{\frac{a}{2^n} - \frac{a}{2^{n-1}}}{r_n},$$

whence
$$\frac{1}{2r_1} = \frac{1}{2^n r_n},$$

and therefore
$$2^{n-1} r_n = r_1;$$

a result which shews that, r_1 being the radius of the second pully, the radii of the rest must be

$$\frac{r_1}{2}, \frac{r_1}{2^2}, \frac{r_1}{2^3}, \frac{r_1}{2^4}, \ldots\ldots$$

(5) In raising the weight (in a system of pullies where each pully hangs by a separate string) two inches, five feet four inches of string pass through the hand: to ascertain the number of the pullies.

If P denote the power and W the weight, then, n denoting the number of moveable pullies,

$$P : W :: 1 : 2^n.$$

Also, α, β, being the spaces through which P, W, respectively move, we have, by the principle of Virtual Velocities,

$$P : W :: \beta : \alpha.$$

But, by the hypothesis,

$$\alpha : \beta :: \overset{\text{ft.}}{5}\ \overset{\text{in.}}{4} : \overset{\text{in.}}{2}$$

$$:: 32 : 1.$$

Hence $$2^n : 1 :: 32 : 1,$$

$$2^n = 2^5,$$

and therefore $$n = 5.$$

(6) Supposing the number of pullies to be 3, and a weight of 8 pounds to be suspended from the lowest pully, and one of 24 pounds from the next, find the power.

The power is equal to a force of 7 pounds.

(7) If there be 5 pullies, find what power will support a ton.

The required power is a force of 70 pounds.

(8) In the system of pullies, in which each hangs by a separate string, and the strings are parallel, if there are six moveable pullies, what weight will balance another of 28 pounds acting as the power?

The required weight is four-fifths of a ton.

(9) What force will support a weight of 60 lbs. by means of a system of pullies, in which each pully hangs by a separate string, consisting of six pullies?

The required force is one of 15 oz.

(10) In a system of four pullies, where each hangs by a separate string, the weight supported is 28 lbs.: what is the power?

The required force is 1 lb. 12 oz.

(11) In a system of five pullies, in which each pully hangs by a separate string, the stress of the string downwards on the lowest block but one is 80 lbs: find the weight supported and the power exerted.

The weight supported is 160 lbs. and the power exerted is 5 lbs.

(12) Supposing a power of 3 pounds to sustain a weight of 48 pounds, find the number of the moveable pullies.

There must be four such pullies.

(13) If there be three pullies, in a system where each hangs by a separate string, and if the weight of each be one pound; find the power which will support a weight of nine pounds.

The required power is a force of two pounds.

(14) In a system, where 3 pullies hang by separate and parallel strings, a weight of 3 pounds is attached to the highest, 4 pounds to the next, and 5 pounds to the lowest pully: find the power required to sustain equilibrium.

The required power is a force of 3 pounds 2 ounces.

(15) In a system of pullies, in which each pully hangs by a separate string, there are three pullies of equal weights; the weight attached to the lowest is 32 lbs. and the power is 11 lbs.: find the weight of each pully.

The required weight is 8 lbs.

(16) If the weights of the pullies, reckoning from the one nearest to W, increase in a geometric progression, the common ratio of which is 2, prove that, Q being the weight of the lowest pully,

$$P = \frac{W}{2^n} + \frac{Q}{3} \cdot (2^n - 2^{-n}).$$

SECT. 5. *Second System of Pullies.*

(1) To find the greatest weight which a force of P pounds can raise, in the system of pullies where the same string passes round all the pullies, supposing the weights of the pullies on the lower block to be as the natural numbers, and the least of them to weigh $\frac{1}{2}P$ pounds.

Let n represent the number of the pullies on the lower block: then, W being the weight raised, we have, observing that there will be $2n$ strings at the lower block,

$$2nP = W + \tfrac{1}{2}P(1 + 2 + 3 + \ldots\ldots + n)$$

$$= W + \tfrac{1}{2}P . \frac{n(n+1)}{2},$$

$$W = \tfrac{1}{4}P . n(7 - n).$$

Substituting successively 1, 2, 3,... for n, we shall see that W will be greatest when n is equal to 3 or 4, its greatest value accordingly being $3P$.

(2) If there be twelve strings at the lower block, in a system of pullies in which the same string passes round all the pullies and in which the parts between the pullies are parallel, find the weight which a power of ten pounds will support, the weights of the pullies being neglected.

The required weight is 120 lbs.

(3) In the second system of pullies, a platform is suspended from the lower block: prove that a man of weight W, standing on the platform, may support himself by exerting on the string a force equal to $\frac{m+1}{n+1} . W$, where n is the number of strings and mW the weight of the platform and lower block together.

SECT. 6. *Third System of Pullies.*

(1) If it is required to raise a weight equal to three times the power, how many strings must there be?

The number of strings must be two.

(2) If the weight of the lowest pully, in that system of pullies in which all the strings, n in number, are attached to the weight, be equal to the power P, of the next lowest, to $3P$, and so on, that of the highest moveable pully being $3^{n-2}P$, prove that P will be to W as 2 to $3^n - 1$.

SECT. 7. *The Screw.*

(1) To determine the direction of application of a given power at the extremity of a given arm, so as to support the greatest weight on a rough vertical screw.

Let W be the weight supported on the screw, when it is on the point of motion, μ the coefficient of friction, α the inclination of the screw to the horizon, P the given power, R the normal reaction exerted by the groove at any point on the thread of the screw. It is evident that, to be most effective, P's direction must be at right angles to the arm: let ϕ be its inclination to the horizon.

Then, resolving vertically, we have

$$W = \Sigma\,(R\cos\alpha + \mu R\sin\alpha) + P\sin\phi,$$
$$= (\cos\alpha + \mu\sin\alpha)\,\Sigma(R) + P\sin\phi,$$

and, taking moments about the axis of the screw, a being the length of the arm and r the radius of the cylinder,

$$P\cos\phi\,.\,a = \Sigma\,(R\sin\alpha\,.\,r - \mu R\cos\alpha\,.\,r)$$
$$= r\,(\sin\alpha - \mu\cos\alpha)\,.\,\Sigma(R).$$

Eliminating $\Sigma\,(R)$ between these two equations, we see that

$$Pa\cos\phi\,(\cos\alpha + \mu\sin\alpha) = r\,(W - P\sin\phi)\,(\sin\alpha - \mu\cos\alpha),$$
$$W = P\left\{\sin\phi + \frac{a}{r}\cos\phi\,\frac{\cos\alpha + \mu\sin\alpha}{\sin\alpha - \mu\cos\alpha}\right\},$$

and therefore, putting $\mu = \tan\epsilon$,

$$W = P\{\sin\phi + \frac{a}{r}\cos\phi\cot(\alpha - \epsilon)\}.$$

Putting $\dfrac{W}{P} = f$, and $\dfrac{a}{r}\cot(\alpha - \epsilon) = g$, we see that

$$f = \sin\phi + g\cos\phi,$$

$$f^2 - 2f\sin\phi + \sin^2\phi = g^2\cos^2\phi,$$
$$(1+g^2)\sin^2\phi - 2f\sin\phi = g^2 - f^2$$
$$\{(1+g^2)\sin\phi - f\}^2 = (g^2 - f^2)(1+g^2) + f^2$$
$$= g^2(1+g^2-f^2).$$

This equation shews that, when f is the greatest possible,

$$f^2 = 1 + g^2,$$

and $$(1+g^2)\sin\phi = f:$$

hence $$(1+g^2)^2\sin^2\phi = 1 + g^2,$$

$$\sin^2\phi = \frac{1}{1+g^2},$$

and therefore

$$\cot\phi = g = \frac{a}{r}\cot(\alpha - \epsilon).$$

(2) Construct a screw such that a force of 112 lbs., acting on an arm of ten times the radius of the screw, may raise 5 tons.

The screw must be so constructed that the vertical distance between two threads may be equal to one tenth of the circumference of a circular section of the cylinder.

(3) If the coefficient of friction of a rough screw be $\frac{1}{2}$, find the least number of turns which may be given to the thread in order that a weight may be supported on the screw without the action of any power, the cylinder being 2 feet in length and 6 inches in circumference.

The required number of turns is 8.

(4) If the inclination of a screw be $\frac{\pi}{4}$, $\tan\epsilon$ the coefficient of friction, P the least horizontal power which will prevent the weight from descending, P' the greatest which can be applied without its rising, prove that

$$\frac{P' - P}{P' + P} = \sin 2\epsilon.$$

SECT. 8. *Common Steelyard.*

(1) Two graduations on the longer arm of a common steelyard are 1 inch apart for a difference of 1 lb: the fulcrum is 3 inches from the point to which the scale is attached, and $9\frac{1}{2}$

inches from the graduation indicating 10 lbs: to determine the moment of the beam.

Let C (fig. 101) be the fulcrum, W the weight of the substance in the scale hanging from the end A, P the moveable weight, suspended from a point E. Let B be the origin of the graduations. Then, as is shewn in Treatises on Statics,

$$P \,.\, BE = W \,.\, AC.$$

Hence, since $BE = 9\frac{1}{2}$ inches when $W = 10$ lbs.,

$$P \,.\, (BC + 9\tfrac{1}{2}) = 10 \times 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, since BE increases by 1 inch when W increases by 1 lb., we have also

$$P \,.\, (BC + 10\tfrac{1}{2}) = 11 \times 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we see that $P = 3$, and therefore

$$P \,.\, BC = \tfrac{3}{2},$$

$P \,.\, BC$ being, as is shewn in Treatises on Statics, equal to the moment of the beam about C.

(2) Shew that, if a common steelyard be constructed with a given rod, the weight of which is inconsiderable compared with that of the sliding weight, the sensibility varies inversely as the sum of the sliding weight and the greatest weight which can be weighed.

CHAPTER X.

MISCELLANEOUS PROBLEMS.

(1) Two small smooth rings, of equal weight, slide on a fixed elliptical wire, of which the axis major is vertical, and are connected by a fine string passing over a smooth peg at the upper focus: prove that the rings will rest in whatever position they may be placed.

If S be the upper focus, AC the semi-axis major, and P, P', simultaneous positions of the rings, their depths below the directrix are equal to

$$\frac{AC}{SC}\cdot SP, \quad \frac{AC}{SC}\cdot SP',$$

and therefore the depth of their centre of gravity below the directrix is equal to

$$\frac{AC}{2SC}\cdot (SP + SP'),$$

which, $SP + SP'$ being the length of the string, is a constant quantity. The system of the two rings is therefore equivalent to that of their two weights condensed at their centre of gravity and supported on a rigid smooth line parallel to the directrix. Thus, as far as equilibrium is concerned, we may conceive PSP' as a weightless triangle, every position which it can assume consistently with the geometry being therefore one of rest.

(2) A certain force will balance two equal forces P, P, acting on a point O, when α is the angle between their directions, and will balance two equal forces Q, Q, when β is the angle between their directions: prove that

$$\frac{P}{Q} = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}.$$

(3) Three forces act on a body along the three sides BC, CA, AB, of a triangle, to which they are respectively proportional: prove that these three forces are equivalent to two equal forces acting, in opposite directions, in parallel lines.

(4) Find the length of a horizontal cylinder of uniform density, every foot of which weighs a pound, when it is balanced by 40 pounds at a distance of one foot from the fulcrum, the distance of the fulcrum from the middle point of the rod being four feet.

The required length is 10 feet.

(5) A sphere, the weight of which is W and radius a, is supported on a plane, the inclination of which is α, by a string fastened to a point in its surface, passing over a small pully at the top of the plane, and supporting a weight P: find the position of equilibrium.

When the sphere is in a position of equilibrium, the distance of its point of contact from the pully is equal to

$$\frac{Wa \sin \alpha}{(P^2 - W^2 \sin^2 \alpha)^{\frac{1}{2}}},$$

which shews that the equilibrium is impossible unless P be greater than $W \sin \alpha$.

(6) A quadrantal arc, without weight, rests with its convex part on a horizontal plane, its axis being vertical: shew that, if two weights P, Q, be placed at the extremities of the arc, it will assume a position of equilibrium, when turned through an angle equal to

$$\tan^{-1} \frac{P \sim Q}{P + Q}.$$

(7) A rope AB, of given length and in a given state of tension, connects a point B of the trunk of a tree CD with a point A in a horizontal line passing through the base C of the tree; find the position of AB in order that the effect of the rope to break the tree about C may be the greatest possible.

The rope must be inclined to the ground at an angle of 45°.

(8) A weight is placed upon a smooth inclined plane: shew that it is impossible for a force, acting at right angles to the plane, to produce equilibrium.

(9) A wheel, moveable about its centre in a vertical plane, has three weights equal to 1, 2, 3, at three equidistant points in the circumference: as it turns round, find the greatest moment of the weights about the centre.

The moment will be greatest whenever the weight 1 is at its greatest distance from a diameter of the wheel which is inclined to the horizon at an angle of 60°.

(10) A rod is supported horizontally by two props, at given distances from its centre of gravity: find the pressure on each.

If a, b, represent the two distances, P, Q, the corresponding pressures, and W the weight of the rod,

$$P = \frac{bW}{a+b}, \quad Q = \frac{aW}{a+b}.$$

(11) ABC is a heavy triangle, of weight 9 lbs. AD bisects BC in D, and DA is produced to F, AF being equal to AD. If AF be a rod without weight, rigidly connected with the triangle ABC, what weight must be suspended at F to balance the triangle about a fulcrum at A?

A weight of 6 pounds.

(12) A uniform rod, attached at its lower end to a smooth hinge on a horizontal plane, leans with its upper end against an inclined plane which rests on the horizontal plane: determine the horizontal force which must be applied to the inclined plane to prevent its sliding.

If α be the inclination of the plane and β of the rod to the horizon, and W the weight of the rod, the required horizontal force is equal to

$$\frac{\frac{1}{2}W}{\cot\alpha + \tan\beta}.$$

(13) If a set of forces, acting on a body along the sides, taken in order, of a plane polygon, be proportional to the sides, shew that their tendency to turn the body about an axis, perpendicular to the plane of the polygon, is the same through whatever point of the plane the axis passes.

(14) An inclined plane, without weight, with a rough base and a smooth inclined face, is set upon a rough horizontal plane: shew that, if the angle of the inclined plane be less than $\tan^{-1}\mu$, μ being the coefficient of friction between the surfaces in contact, no force, applied to the inclined face, will be able to move the inclined plane.

(15) A string $ABCDEP$, fig. (102), is attached to the centre A of a pully, the radius of which is r; it then passes over a fixed point B, and under the pully, which it touches in the points C and D; it afterwards passes over a fixed point E, and has a weight P attached to its extremity; BE is horizontal and equal to $\frac{5}{3}r$, and DE is vertical: supposing the system to be in equilibrium, find the weight of the pully and the distance AB.

The weight of the pully $= \frac{5}{2}P$ and $AB = \frac{8r}{3\sqrt{7}}$.

(16) An endless string supports a system of equal heavy pullies, the highest one of which is fixed, the string passing round every pully and crossing itself between each. If $\alpha, \beta, \gamma, \ldots$ be the inclinations to the vertical of the successive rectilinear portions of the string, prove that $\cos\alpha$, $\cos\beta$, $\cos\gamma, \ldots$ are in arithmetical progression.

(17) Three uniform rods, rigidly connected in the form of a triangle, rest on a smooth sphere of radius r: prove that the inclination of the plane of the triangle to the horizon is equal to $\tan^{-1}\left(\frac{d}{r}\right)$, where d is the distance between the centres of the circles inscribed in the triangle itself and in the triangle formed by joining the middle points of the rods.

(18) A uniform rod is held at a given inclination to a rough horizontal table, by a string attached to one of its ends, the other end resting on the table: find the greatest angle at which the string can be inclined to the vertical, without causing the end of the rod to slide along the table.

If α be the inclination of the rod to the table, and $\tan \epsilon$ the coefficient of friction, the required angle is equal to

$$\cot^{-1} (\cot \epsilon + 2 \tan \alpha).$$

(19) A picture is hung up against a rough vertical wall by a string fastened to a point in its back, so that the picture inclines forwards: apply the principle of the triangle of forces to find the inclination of the string to the wall, when its tension is the least possible.

If $\tan \epsilon$ be the coefficient of friction, the required inclination is equal to ϵ.

(20) A winch handle, by describing a circle, turns a small cog-wheel, which turns the wheel of a Wheel and Axle; the weight to be raised is attached to a single moveable pully; one portion of the string supporting the pully is wound round the axle, and the other is tied to a fixed bar, the strings being parallel; find the mechanical advantage in terms of the radii of the circle described by the winch handle, and of the axle, and of the number of teeth in the cog-wheel, and in the wheel of the Wheel and Axle.

If r be the radius of the circle described by the winch handle, r' the radius of the axle, n the number of teeth in the wheel of the Wheel and Axle, and n' the number in the cog-wheel, the mechanical advantage is equal to

$$\frac{2nr}{n'r'}.$$

DYNAMICS.

CHAPTER I.

FREE RECTILINEAR MOTION.

SECT. 1. *Uniform motion.*

(1) Two candles, which will burn for four and six hours respectively, are placed in candlesticks a foot high and a foot apart, and are lighted at the same moment. The shadow of the shorter is received on the table on which the candlesticks stand; that of the extremity of the longer on a vertical wall, ten feet distant from it, and perpendicular to the plane of the candles. Supposing each candle to be originally a foot long, to find the velocity of the extremity of the shadow of the longer: to find also the mean velocity of the extremity of the shadow of the shorter, during the last hour in which it is burning.

Let AB, $A'B'$, fig. (103), be the candlesticks, BC, $B'C'$, the lengths of the candles at the end of t hours from the moment of lighting, and K, L, the extremities of the shadows of the longer and shorter candles respectively.

Let $BC = x$, $B'C' = y$, $AL = a$, $HK = b$. Then

$$x = 1 - \frac{1}{4}t, \quad y = 1 - \frac{1}{6}t.$$

Also, by similar triangles,

$$\frac{b - y - 1}{10} = \frac{y + 1}{a + 1} = \frac{x + 1}{a}.$$

Hence, substituting for x, y, their respective values in terms of t, we have

$$\frac{12-t}{3a+3} = \frac{8-t}{2a} \quad\text{.......................(1)},$$

and
$$\frac{6b+t-12}{15} = \frac{8-t}{a} \quad\text{.......................(2)}.$$

From (1) we see that

$$24a - 2at = 24a - 3at + 24 - 3t,$$

and therefore
$$a = \frac{24}{t} - 3 \quad\text{.........................(3)}.$$

From (2) and (3) there is

$$\frac{6b+t-12}{15} = \frac{t(8-t)}{3(8-t)} = \frac{t}{3},$$

$$6b + t - 12 = 5t,$$

$$b = 2 + \frac{2}{3}t \quad\text{.......................(4)}.$$

The equation (3) shews that $a = 5$ when $t = 3$, and that $a = 3$ when $t = 4$: hence the extremity of the shadow of the shorter candle describes 2 feet in the last hour: its mean velocity is therefore, during the last hour, 2 feet per hour.

Again, the equation (4) shews that the extremity of the shadow of the longer candle has a uniform velocity of 8 inches per hour.

(2) One body moves through 40 feet in 3 seconds, and another through 25 yards in 6 minutes: compare their velocities.

The velocity of the former body is 64 times as great as that of the latter.

(3) What is the ratio of the velocity of light to that of a cannon ball, which issues from a gun with a velocity of 1500 feet per second, light passing from the Sun to the Earth, a distance of ninety-five millions of miles, in $8\frac{1}{3}$ minutes?

The required ratio is 668800.

(4) A person travelling eastward, at the rate of 4 miles an hour, observes that the wind seems to blow directly from the north, and that, on doubling his speed, it appears to come from the north-east: determine the direction of the wind, and its velocity.

The wind blows from the north-west with a velocity of $4\sqrt{2}$ miles an hour.

(5) Two bodies move uniformly along two straight lines from their point of intersection, their velocities being inversely proportional to their masses: shew that their centre of gravity describes the line bisecting the angle between them, and determine its velocity.

If u, v, denote the velocities of the two bodies, and ω the angle between their paths, the required velocity is equal to

$$\frac{2\cos\frac{\omega}{2}}{\frac{1}{u}+\frac{1}{v}}.$$

(6) ABC is a triangle: two spheres start together from A, B, their centres moving along AC, BC, with velocities which would carry them separately to C in the same time; find the distance each has gone through when they meet.

If a, b, c, be the sides of the triangle, and r, s, the respective radii of the spheres, the required distances are equal to

$$\frac{b}{c}(c-r-s), \quad \frac{a}{c}(c-r-s).$$

SECT. 2. *Acceleration.*

(1) Through how many inches does a heavy particle, let fall from rest, descend in the first half-second of its motion? If it were to move uniformly during the next half-second with the velocity thus acquired, through what space would it move during that interval?

Putting $t=\frac{1}{2}$, and $g=12\times 32{\cdot}2$, in the formula $s=\frac{1}{2}gt^2$, we see that the number of inches in the descent is equal to

$$\frac{1}{2}\times 12\times 32{\cdot}2\times\frac{1}{4}=3\times 16{\cdot}1=48{\cdot}3.$$

The space which would be described by the particle in one second, with the velocity acquired in one second, is equal to 32·2 feet: hence the space which would be described in one second, with the velocity acquired in half a second, is equal to 16·1 feet, and therefore the space which would be described in half a second, with the velocity acquired in half a second, is equal to 8·05 feet.

(2) If a body be projected vertically upwards, with a velocity $8g$, to find the time in which it will rise through the height $14g$.

Let t denote the required time: then

$$14g=8gt-\frac{1}{2}gt^2,$$

$$t^2-16t=-28,$$

$$t^2-16t+64=36,$$

$$t=8\pm 6,$$

$$t=2 \text{ or } =14.$$

Hence the body will arrive at the height $14g$ after 2 seconds, and, after reaching its greatest altitude, will descend to the same point, the whole interval between its projection and second arrival at this point being 14 seconds.

(3) A body, falling to the ground, is observed to pass through $\frac{8}{9}$ths of its original height in the last second: find the height.

Let $h=$ the original height, and $t=$ the whole time of the descent: then

$$t=\left(\frac{2h}{g}\right)^{\frac{1}{2}},\quad t-1=\left(\frac{2h}{9g}\right)^{\frac{1}{2}},$$

and therefore

$$\left(\frac{2h}{g}\right)^{\frac{1}{2}} - \frac{1}{3}\left(\frac{2h}{g}\right)^{\frac{1}{2}} = 1,$$

$$2\left(\frac{2h}{g}\right)^{\frac{1}{2}} = 3,$$

$$h = \frac{9}{8}g = 36 \text{ feet nearly.}$$

(4) A body, falling in vacuum under the action of gravity, is observed to describe 144·9 feet and 177·1 feet in two successive seconds; to determine the accelerating force of gravity, and the time from the beginning of the motion.

Let u represent the velocity of the body at the beginning of the two seconds during which its motion is observed: then, by the formula for falling bodies,

$$s = Vt + \tfrac{1}{2}gt^2,$$

putting $t = 1$, $V = u$, and $s = 144{\cdot}9$, we have

$$144{\cdot}9 = u + \tfrac{1}{2}g \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, at the beginning of the second of the two seconds, the velocity of the body is $u + g$: hence, putting, in the standard formula, $t = 1$, $s = 177{\cdot}1$, and $V = u + g$, we obtain

$$177{\cdot}1 = u + g + \tfrac{1}{2}g \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Subtracting (1) from (2), we get

$$g = 32{\cdot}2,$$

and therefore, by (1),

$$u = 144{\cdot}9 - 16{\cdot}1 = 128{\cdot}8.$$

Let t denote the number of seconds from the beginning of the motion to the beginning of the first of the two seconds: then

$$t = \frac{u}{g} = \frac{128{\cdot}8}{32{\cdot}2} = 4.$$

(5) A constant force acts upon a body from rest during 3 seconds and then ceases: in the next 3 seconds it is found that the body describes 180 feet: to find both the velocity of

the body at the end of the 2nd second of its motion and the numerical values of the accelerating force, (1) when a second, (2) when a minute is taken as the unit of time.

Let V denote the velocity acquired at the end of 3 seconds, and f the accelerating force, a second being taken as the unit of time.

Then $$3V = 180, \quad V = 3f,$$

and therefore $$f = 20 \text{ feet}.$$

Also the velocity, at the end of two seconds, is equal to $2f = 40$ feet.

The velocity which would be acquired from rest in one minute, under the action of the accelerating force, is equal to $60f = 1200$ feet: hence the body would describe in one minute, with the velocity acquired in one minute, a space equal to 72000 feet. Therefore, if a minute instead of a second be taken as the unit of time, the accelerating force will be equal to 72000 feet.

(6) A particle descends by gravity and describes, in the n^{th} second of its fall, a space equal to p times the space described in the last but n: to find the whole space fallen through from rest.

Let s = the whole space fallen through and t = the whole number of seconds.

The velocity acquired at the beginning of the n^{th} second is $(n-1)g$: hence the space described in the n^{th} second is equal to

$$(n-1)g + \tfrac{1}{2}g = \tfrac{1}{2}(2n-1)g.$$

Again, the velocity acquired at the beginning of the last second but n, that is, at the end of the $(t-n-1)^{\text{th}}$ second, is equal to $(t-n-1)g$, and, accordingly, the space described, in the last second but n, is equal to

$$(t-n-1)g + \tfrac{1}{2}g = \tfrac{1}{2}(2t-2n-1)g.$$

Hence, by the hypothesis,

$$\tfrac{1}{2}(2n-1)g = \tfrac{1}{2}p(2t-2n-1)g,$$

$$2pt = 2n - 1 + (2n+1)p;$$

and therefore

$$s = \tfrac{1}{2}gt^2 = \frac{g}{8p^2}\{2n - 1 + p(2n+1)\}^2.$$

(7) A body is projected vertically upwards with a velocity $4g$: after two seconds suppose gravity to cease to act for one second, and then to be doubled: to find the greatest height to which the body ascends, and the velocity when it returns to the point of projection.

At the end of two seconds the body has an upward velocity equal to $4g - 2g = 2g$, its height being equal to

$$4g \times 2 - \tfrac{1}{2}g \times 4 = 6g.$$

At the end of three seconds, it still has an upward velocity $2g$, its height being equal to

$$6g + 2g = 8g.$$

If h denote the additional height acquired, before the motion ceases,

$$(2g)^2 = 2 \times 2g \times h,$$

whence $$h = g.$$

Thus the greatest altitude at which the body arrives, above the point of projection, is equal to

$$6g + 2g + g = 9g.$$

Let V denote its velocity, when it again reaches the point of projection: then

$$V^2 = 2 \times 2g \times 9g,$$

and therefore $$V = 6g.$$

(8) A particle, projected in the direction of a uniform force with a velocity u, after describing a space s acquires an additional velocity v: it acquires a second additional velocity v, after describing an additional space $2s$: to find the ratio between u and v.

We have $$(u+v)^2 = u^2 + 2fs,$$

and $$(u+2v)^2 = u^2 + 6fs,$$

and therefore

$$3(u+v)^2 - (u+2v)^2 = 2u^2,$$

$$2uv - v^2 = 0,$$

whence $$u : v :: 1 : 2.$$

(9) A heavy body has fallen from A to B, when another body is let fall from C, a point lower than B in the same vertical line: how far will the latter body fall before it is overtaken by the former?

Let $AB = a$, $BC = b$. Then, at the instant when the latter body is let go, the velocity of the former body is $(2ga)^{\frac{1}{2}}$. Since gravity cannot affect the relative velocity of the two bodies, $(2ga)^{\frac{1}{2}}$ is therefore always their relative velocity: hence, b being their initial distance, and t the time between the commencement of the second body's motion and the collision of the two bodies,

$$t = \frac{b}{(2ga)^{\frac{1}{2}}}.$$

Hence the space fallen through by the latter body is equal to

$$\tfrac{1}{2}gt^2 = \frac{b^2}{4a}.$$

The solution may be effected also in the following manner. Let s be the required fall of the latter body: then

$$s = \tfrac{1}{2}gt^2,$$

and, $b + s$ being the space described by the former in the time t,

$$b + s = (2ga)^{\frac{1}{2}}t + \tfrac{1}{2}gt^2,$$

and therefore $$b = (2ga)^{\frac{1}{2}}t, \quad s = \frac{b^2}{4a}.$$

(10) A body, moving in a straight line, has velocities v_1, v_2, v_3, ... after intervals of one second each: in what case is the force uniform, and what is then its measure?

The velocities v_1, v_2, v_3 ... must form an arithmetical progression, the measure of the force being the common difference.

(11) If a heavy body fall from rest through 144 feet, determine the time of the motion.

The required time is approximately equal to 3 seconds.

(12) A body, dropped from the top of a tower, the height of which is 60 feet, reaches the bottom of a well, at the foot of the tower, in 3″: find the depth of the well.

The depth of the well is 84·9 feet.

(13) Determine the height through which a body will fall in two seconds and a half, and the velocity acquired.

The required height and velocity are respectively, in feet, 100·625 and 80·5.

(14) A, B, C, D are points in a vertical line, the lengths AB, BC, CD being equal: if a body falls from A, prove that the times of describing AB, BC, CD are respectively as

$$1 : \sqrt{2}-1 : \sqrt{3}-\sqrt{2}.$$

(15) A body describes in successive intervals, of 4 seconds each, the spaces 24 and 64 feet, in the same straight line: determine the accelerating force and the velocity at the beginning of the first interval.

The measure of the accelerating force is 2 feet 6 inches, and the measure of the required velocity is one foot.

(16) A particle moves over 7 feet, in the first second of the time during which it is observed, and over 11 and 17 feet in the third and sixth seconds respectively: prove that these facts are consistent with the supposition of its being subject to the action of a uniform force.

(17) A body, starting with a given velocity, moves for a given time under the action of a uniform force in the direction of its motion: shew that an equal space would be described by a body moving uniformly during the given time with a velocity equal to half the sum of the initial and final velocities of the first body.

(18) A falling body is observed, at one portion of its path, to pass through n feet in r seconds: find the number of feet described in the next r seconds.

The required number of feet is equal to $n + gr^2$.

(19) A falling body has a velocity u, at first, and a velocity v, at the end of t seconds: shew that the space described is equal to

$$\tfrac{1}{2}t\,(u+v).$$

(20) If s, ms, be the spaces described by a body in times t, nt, respectively, determine the magnitude of the accelerating force and the velocity of projection.

The accelerating force and velocity of projection are respectively equal to

$$\frac{2(m-n)}{n(n-1)} \cdot \frac{s}{t^2} \text{ and } \frac{m-n^2}{n(1-n)} \cdot \frac{s}{t}.$$

(21) If the number of units of space, described by a body in the last second of its fall, be to the number of units in the final velocity, as 8 to 9, for how many seconds does the body fall?

The time of falling is $4\frac{1}{2}$ seconds.

(22) A body, acted on by a uniform force, has described 100 feet from rest in $2'$: in what time will it pass over the next 125 feet?

The required time is one second.

(23) Supposing gravity to act on a body during the 1st, 3rd, 5th, &c., seconds and not during the 2nd, 4th, &c.; shew that the space described from rest in $2t$ seconds is equal to

$$\tfrac{1}{2}gt(2t+1).$$

(24) A stone, dropped into a well, is heard to strike the water after t seconds: find the depth of the surface of the water, the velocity of sound being assumed as known.

If $u =$ the velocity of sound, the required depth is equal to

$$\left\{\left(\frac{u^2}{2g}+ut\right)^{\frac{1}{2}} - \frac{u}{(2g)^{\frac{1}{2}}}\right\}^2.$$

(25) The velocity of a body increases from ten to sixteen feet per second, in passing over thirteen feet under the action of a constant force; find the numerical value of the force.

The numerical value is 6.

(26) Since $(2fs)^{\frac{1}{2}}$ is the velocity generated by an accelerating force f in a body moving through a space s, when there is no initial velocity, therefore, by the second law of motion, $u+(2fs)^{\frac{1}{2}}$ is the velocity, after motion through s, when there is an initial velocity u. Point out the fallacy in this argument.

(27) A body is projected vertically upwards with a velocity of 25 feet: determine its height and velocity at the end of two seconds.

At the end of 2 seconds the body is descending with a velocity of 39·4 feet, and its depth below the point of projection is 14·4 feet.

(28) A body is projected vertically upwards with a velocity of a hundred feet: determine its altitude of ascent at the end of two seconds.

The required altitude is equal to 135·6 feet.

(29) A body is projected vertically upwards with a velocity of a hundred feet: determine the greatest height to which it will ascend, and the time of ascending to this height.

The required height is 155·28 feet and the required time of ascent is 3·1 seconds, approximately.

(30) A body is projected vertically upwards with a velocity of $3g$ feet: find its height and velocity at the end of four seconds.

The required height is $4g$ feet, and its velocity, which is downward, is one of g feet.

(31) A body is thrown vertically upwards with a velocity $3g$: at what times will its height be $4g$, and what will be its velocity at these times?

It will be at the height $4g$, first, at the end of two seconds, and, again, at the end of four seconds: its velocity at both these instants is g, being upward at the end of two seconds and downward at the end of 4.

(32) A body is projected vertically upwards with a velocity which will carry it to a height of $2g$ feet: after how long a time will it be descending with a velocity g?

After an interval of 3 seconds.

(33) It is said that, on one of the asteroids, a man, who on the Earth could leap a height of 6 feet, could jump 60 feet high: compare the attraction of the asteroid on its surface with the force of terrestrial gravitation.

The Earth's attraction is ten times as great as that of the asteroid.

CHAPTER II.

PROJECTILES.

SECT. 1. *Motion resolved horizontally and vertically.*

(1) To prove that the time of describing any portion PQ of the parabolic path of a body, acted on by gravity, is proportional to the difference of the tangents of the angles which the tangents at P and Q make with the horizon.

Let V, V' be the velocities of the body at P, Q, respectively, α, α' being the inclinations of the tangents at these points to the horizon. Let $u =$ the horizontal velocity of the body, and t the time of describing the arc PQ.

Then $$V \cos \alpha = u = V' \cos \alpha',$$

and $$V' \sin \alpha' = V \sin \alpha - gt.$$

From these equations we see that

$$t = \frac{u}{g} (\tan \alpha - \tan \alpha'):$$

hence $$t \propto \tan \alpha - \tan \alpha'.$$

(2) Having given the velocities at two points of the path of a projectile, to find the difference of their altitudes above a horizontal plane.

Let u, v be the horizontal and vertical components of the velocity at any point A of the path: v' being the vertical component at a point P, h feet higher than A. The horizontal component will be the same at both points.

Now $$v'^2 = v^2 - 2gh,$$

and, if V be the whole velocity at P,

$$V^2 = u^2 + v'^2:$$

hence $$V^2 = u^2 + v^2 - 2gh.$$

Similarly, V' being the velocity at a point P', h' feet higher than A,

$$V'^2 = u^2 + v^2 - 2gh'.$$

Hence $$V^2 \sim V'^2 = 2g\,(h' \sim h),$$

and therefore the difference of the altitudes of P, P', above a horizontal plane, is equal to

$$\frac{V^2 \sim V'^2}{2g}.$$

(3) If the focus of the path of a projectile be as much below the horizontal plane through the point of projection, as the highest point of the path is above it; to find the angle of projection.

If V be the velocity and α the angle of projection, the greatest altitude is equal to

$$\frac{V^2 \sin^2 \alpha}{2g}.$$

Also the distance between the vertex of the path and the focus is equal to a quarter of the latus rectum, that is, to

$$\frac{V^2 \cos^2 \alpha}{2g}.$$

Hence, by the hypothesis,

$$\frac{V^2 \sin^2 \alpha}{2g} = \frac{1}{2} \cdot \frac{V^2 \cos^2 \alpha}{2g},$$

whence $$\tan^2 \alpha = \frac{1}{2},$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

(4) Particles are projected from the same point in the same direction, but with different velocities; to find the locus of the foci of their paths.

Let O, fig. (104), be the point of projection, A the vertex of the parabolic path, S its focus, OH half the horizontal range.

Join OS. Let $V =$ the velocity and $\alpha =$ the angle of projection. Then

$$AS = \frac{V^2 \cos^2 \alpha}{2g},$$

$$AH = \frac{V^2 \sin^2 \alpha}{2g},$$

and therefore
$$HS = \frac{V^2}{2g} \cos 2\alpha.$$

Also
$$OH = \frac{V^2 \sin 2\alpha}{2g}.$$

Hence, θ denoting the angle SOH,

$$\tan \theta = \cot 2\alpha.$$

This result shews that the locus of S is a straight line through O making an angle with OH equal to $\frac{1}{2}\pi - 2\alpha$.

(5) If a body be projected in a direction inclined to the horizon, to prove that the time of moving between two points at the extremities of a focal chord of the parabolic path is proportional to the product of the velocities of the body at the two points.

Let P, P', fig. (105), be the ends of any focal chord PSP': draw $PH, P'H', SK$, vertically, to meet the directrix in H, H', K. Let $\angle PSK = \theta$, $t =$ the time from P to P', $u =$ the horizontal component of the body's velocity. Also, let V, V', denote the velocities at P, P'. Then, since the velocity at any point of the path is that due to its depth below the directrix,

$$V^2 = 2g \,.\, PH, \quad V'^2 = 2g \,.\, P'H',$$

$$(V.V')^2 = 4g^2.PH.P'H'.$$

But
$$HP + SP \cos \theta = SK = H'P' - SP' \cos \theta,$$

and therefore, since $SP = HP$, and $SP' = H'P'$,

$$HP = \frac{SK}{1 + \cos \theta}, \quad H'P' = \frac{SK}{1 - \cos \theta},$$

and therefore

$$V.V' = \frac{2g \,.\, SK}{\sin\theta} \text{ .... } (1).$$

Again, $PP' \sin\theta$ being the horizontal motion of the body in the time t,

$$u\,.\,t = PP' \sin\theta = (HP + H'P') \sin\theta = \frac{2SK}{\sin\theta} \text{ ...... } (2).$$

From (1) and (2) we see that $t \propto V.V'$.

(6) Several bodies are projected at the same instant from a point in different directions with the same velocity: to find their locus at the end of a given time.

Since gravity acts in the same direction and with the same intensity upon all the bodies, their relative motion will not be affected by it. But, supposing gravity not to exist, it is plain that the locus of the bodies would be a sphere, its centre being the point of projection. Hence, under the actual circumstances of the case, the locus of the bodies will be a sphere, the radius of which varies as the time and of which the centre descends according to the law of a falling body.

(7) Two heavy particles are projected from two given points in the same vertical line in parallel directions and with equal velocities: prove that tangents, drawn to the path of the lower, will cut off, from the path of the upper, arcs described in equal times.

Let a vertical line intersect the upper and lower parabolic paths in P, V, (fig. 106), respectively, and let QVQ' be a chord of the upper touching the lower in V. Since the two paths are evidently similar and similarly placed and since their vertices must evidently be in the same vertical line, it is clear that QVQ' must be parallel to the tangent to the upper parabola at P. It is also clear that PV must be always equal to the distance between the two vertices, that is, constant. Draw QD at right angles to PV, produced if necessary. Then, by a property of the parabola,

$$QD^2 \propto PV.$$

Hence QD is constant; and therefore the horizontal distance between Q, Q', which is equal to $2QD$, is constant: but the horizontal velocity of the projectile is constant: hence the time through the arc QQ' is invariable.

(8) A body is projected vertically upwards from a point A with a given velocity: find the direction in which another body must be projected with a given velocity from a point B, in the same horizontal line with A, so as to strike the first body.

If V and V' denote the projectile velocities of the former and latter body respectively, and θ the inclination of V' to the horizon,

$$\sin\theta = \frac{V}{V'}.$$

(9) A body is projected with a horizontal velocity of ·8 feet, and a vertical velocity of 16·7 feet: prove that its distance from the point of projection, at the end of one second, is one foot, supposing g to be equal to 32·2 feet.

(10) An arrow, shot vertically upwards, attains a height of 200 feet: find the greatest horizontal distance the arrow may be shot with the same force.

The required distance is 400 feet.

(11) Given the time of flight of a projectile on a horizontal plane, find the greatest height to which it rises.

If t denote the time of flight, the required height is equal to $\frac{1}{8}gt^2$.

(12) Determine the directions in which a particle must be projected with a given velocity, in order that it may hit a mark at a given distance on the same horizontal line with the point of projection.

If $a =$ the given distance, and $V =$ the given velocity, there are two complementary angles of projection, of which the values are

$$\tfrac{1}{2}\sin^{-1}\left(\frac{ag}{V^2}\right) \text{ and } \tfrac{1}{2}\left\{\pi - \sin^{-1}\left(\frac{ag}{V^2}\right)\right\}.$$

(13) If two bodies be projected from the same point, at the same instant, with velocities, the horizontal components of which are u, u', and the vertical components respectively v, v', prove that the time which elapses between their transits through another point, which is common to both their paths, is equal to

$$\frac{2}{g} \cdot \frac{uv' \sim u'v}{u + u'}.$$

(14) If α be the angle of projection of a projectile, T the time which elapses before the body strikes the ground, prove that, at the time $\frac{T}{4 \sin^2 \alpha}$, the angle, which the direction of motion makes with the direction of projection, is equal to $\frac{\pi}{2} - \alpha$.

(15) A stone, thrown at an elevation of 45^0 from the top of a tower, fell in 4 seconds at a distance of 60 feet from the base: find the height of the tower. If the stone had been thrown horizontally, in what time would it have fallen to the ground?

The height of the tower is 197·6 feet. If the stone had been thrown horizontally, the required time would have been approximately $3\frac{1}{2}$ seconds.

(16) A stone, thrown obliquely with a velocity V, reaches a height h: determine the velocity which it has at the highest point.

The required velocity is equal to

$$(V^2 - 2gh)^{\frac{1}{2}}.$$

(17) A body is projected horizontally, with a velocity $4g$, from a point, the height of which above the ground is $16g$: find the direction of motion, (1), when it has fallen halfway to the ground, (2), when half the whole time of falling has elapsed.

The inclinations of the motion to the horizon at the two instants are respectively $\frac{\pi}{4}$ and $\tan^{-1}\left(\sqrt{2}\right)$.

(18) If R, R', be the ranges of two projectiles, which, being thrown from the same place, attain the same vertical height, and pass through a common point, prove that

$$R \cdot R' = \frac{4H \cdot h^2}{k};$$

where H is the greatest height attained, k the height of the common point, and h the horizontal distance of the point of projection from the vertical line through the common point.

(19) A body is projected from a certain point vertically downwards: at the same moment another body is projected from another point with the same velocity as the former body: find the direction of projection of the latter body in order that it may strike the former.

If α be the inclination of the distance between the two points to the horizon, the inclination of the direction of projection of the latter body to the horizon must be equal to $\frac{1}{2}\pi - 2\alpha$.

(20) If V, V', V'', be the velocities at three points P, Q, R, of the path of a projectile, where the inclinations to the horizon are α, $\alpha - \beta$, $\alpha - 2\beta$, and if t, t', be the times of describing PQ, QR, respectively, prove that

$$V''t = Vt', \text{ and } \frac{1}{V} + \frac{1}{V''} = \frac{2 \cos \beta}{V'}.$$

(21) A body is projected from a given point and strikes another given point: supposing this to be possible for only one angle of projection, determine the velocity of projection.

If b be the altitude of the latter point above the former, and c the distance between them, the required velocity is equal to

$$\{(b + c)\, g\}^{\frac{1}{2}}.$$

(22) If a body be projected with a velocity V, in a direction making an angle α with the horizon, shew that, at a height h,

the direction of its motion will make with the horizon an angle equal to

$$\tan^{-1}\left(\tan^2\alpha - \frac{2gh}{V^2}\sec^2\alpha\right)^{\frac{1}{2}}.$$

(23) If r_1, r_2, r_3, be the distances of a projectile from the point of projection, when its angular elevations above this point are respectively α_1, α_2, α_3, prove that

$$r_1\cos^2\alpha_1\sin(\alpha_2-\alpha_3)+r_2\cos^2\alpha_2\sin(\alpha_3-\alpha_1)+r_3\cos^2\alpha_3\sin(\alpha_1-\alpha_2)=0.$$

(24) A cannon is pointed in a direction making an angle of 30° with the horizontal plane on which it stands, and fired against a fort: it is then drawn $\frac{3}{4}$ of a mile nearer the fort, and pointed at the same elevation to the horizon as before, when it is observed that the ball strikes the fort in the same point as in the former case. If the greatest distance which the cannon can throw the ball is one mile, prove that the height of the point which the ball strikes is 165 feet above the horizontal plane on which the cannon stands.

(25) A body is projected, at an angle α to the horizon, so as just to clear two walls of equal height a, at a distance $2a$ from one another: prove that the range is equal to

$$2a\cot\frac{\alpha}{2}.$$

(26) A body is thrown over a triangle, passing from one extremity of the horizontal base, just over the vertex, to the other end of the base: prove that

$$\tan\theta = \tan\alpha + \tan\beta,$$

where θ is the angle of projection, and α, β, are the angles at the base of the triangle.

(27) It is required to throw a shell from a point at a distance a from the foot of a wall, so that it may just clear the top of the wall, the height of which is h, and strike the ground, which is horizontal, at a distance b beyond the wall. Determine

the velocity and angle of projection, neglecting the resistance of the air.

If α denote the angle and V the velocity of projection,

$$\tan \alpha = \frac{h}{ab}(a+b), \quad V^2 = \frac{g}{2abh}\{a^2b^2 + h^2(a+b)^2\}.$$

(28) If h_1, h_2, h_3, be the heights of the sights of a rifle, when adapted for shooting at the distances 100, 200, and 400 yards, respectively; prove that

$$4h_1(h_2^2 - h_3^2) + 2h_2(h_3^2 - h_1^2) + h_3(h_1^2 - h_2^2) = 0.$$

(29) A body is projected from the deck of a ship, with a velocity V, relatively to the ship, and at an angle of elevation β, so as to hit a mark in the wake of the ship and in the horizontal plane of the deck: the ship is supposed to be sailing in a straight course with a velocity V'. Shew that, if α be the angle of projection in order that the body, propelled by the same force, might have hit the same mark had the vessel been at rest, then

$$\frac{V}{V'} = \frac{2\sin\beta}{\sin 2\beta - \sin 2\alpha}.$$

(30) Determine the angle of elevation at which a body must be projected in order that the focus of its path may lie in the horizontal plane passing through the point of projection.

The angle of projection must be 45°.

(31) If any number of bodies be projected in different directions from the same point with equal velocities, find the locus of the foci of the parabolas.

If V denote the velocity of projection, the locus will be a sphere, of which the centre is the point of projection and radius $\frac{V^2}{2g}$.

(32) A number of heavy particles are projected in any directions in a vertical plane from the same point, (1), with the same vertical velocity, (2), with the same horizontal velocity:

prove that in each case the locus of the foci of their paths is a parabola with its focus at the point of projection and axis vertical, but that, in the first case, the vertex is upwards, and, in the second case, downwards.

(33) A particle is projected horizontally with a velocity of $32\sqrt{3}$ feet per second: after a certain time, under the action of gravity, it has a velocity of 64 feet: find this time: determine also the spaces described, horizontally and vertically, and the latus rectum of the curve, supposing the force of gravity to be measured by 32 feet.

The required time is one second, the horizontal and vertical spaces described, $32\sqrt{3}$ and 16 feet respectively, and the latus rectum of the parabola 192 feet.

(34) Heavy particles are projected horizontally with different velocities from the same point; shew that the extremities of the latera recta of the parabolas, which they severally describe, lie on a cone, of which the axis is vertical, and the vertical angle $2\tan^{-1}2$.

(35) Swift of foot was Hiawatha;
He could shoot an arrow from him,
And run forward with such fleetness,
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward,
Shoot them with such strength and swiftness,
That the tenth had left the bow-string
Ere the first to Earth had fallen.

Supposing Hiawatha to have been able to shoot an arrow every second, and, when not shooting vertically, to have aimed so that the flight of the arrow might have the longest range, prove that it would have been safe to bet long odds on him if entered for the Derby.

SECT. 2. *Motion resolved parallel to any straight lines.*

(1) If a body be projected from a given point so as to strike an inclined plane through that point at right angles, to prove that

$$\tan\theta = \frac{1}{2}\cot\alpha,$$

where θ is the angle which the direction of projection makes with the plane, and α the inclination of the plane to the horizon.

The component of the velocity of projection, at right angles to the plane, is $V\sin\theta$, and the component of gravity, at right angles to the plane, is $g\cos\alpha$: hence, t being the time of flight,

$$t = \frac{2V\sin\theta}{g\cos\alpha} \quad \text{..........................} \quad (1).$$

Again, the velocity parallel to the plane being, by the hypothesis, zero at the time of impact, the projectile velocity $V\cos\alpha$, parallel to the plane, must have been destroyed by the component of gravity $g\sin\alpha$ in the time t: hence

$$V\cos\theta = g\sin\alpha\,.\,t \quad \text{.....................} \quad (2).$$

From (1) and (2) it is plain that

$$\tan\theta = \frac{1}{2}\cot\alpha.$$

(2) A body is projected from a given point with a given velocity: to find the direction of projection in order that its path may touch a given plane.

Let A, fig. (107), be the point of projection, BC the given plane: draw AB horizontally to meet BC in B, and draw AD at right angles to BC.

Let $AB = a$, $\angle ABC = \beta$; let V denote the velocity of projection and θ the inclination of this velocity to AD.

Then, since the velocity normal to BC must, by the nature of the question, be zero when the body reaches BC, we have

$$(V\cos\theta)^2 = 2g\cos\beta \,.\, AD$$

$$= 2g\cos\beta \,.\, a\sin\beta,$$

$$\cos\theta = \frac{(ga\sin 2\beta)^{\frac{1}{2}}}{V}.$$

This value of $\cos\theta$ gives two equal values of θ with opposite signs, shewing that two directions of projection will satisfy the conditions of the problem, these directions making equal angles with AD on opposite sides of this line.

(3) From several points of an inclined plane bodies are projected simultaneously in different directions, in such a manner that the times of flight above the plane are the same: to prove that the locus of the bodies, at any moment, is a plane parallel to the inclined plane.

Since the times of flight are the same, the projectile velocities, normal to the inclined plane, must also be the same, and therefore the distances of the bodies from the inclined plane at any time must be the same, that is, their locus at any time must be a plane parallel to the inclined plane.

(4) Two balls are shot at the same moment, from given points, straight at the same mark: to compare their initial velocities, so that they may hit each other.

Let P, P', be the two given points, and C the mark. Also let V, V', represent the velocities of projection.

Supposing gravity not to act, the two bodies would hit each other at C, provided that

$$\frac{V'}{V} = \frac{CP'}{CP},$$

and not otherwise.

But gravity does not affect the relative motion of the bodies: hence they will not hit each other, under the actual circumstances of the case, unless this same relation between V and V' is established.

(5) To find the least velocity with which a body can be projected from a given point so as to hit a given mark, and the direction of projection in this case.

Let A, fig. (108), be the given point, B the given mark. Draw AC horizontally to meet the vertical line BC in C, and produce CB to meet AD, the direction of projection, in D.

Let $AC = a$, $BC = b$, $AB = c$, $\angle BAC = \beta$, $\angle CAD = \theta$, $V =$ the velocity of projection, $t =$ the time of motion from A to B.

Then, resolving the motion parallel to AD and vertically, we have

$$AD = Vt,$$

and therefore
$$a = Vt \cos\theta \quad\text{.................... (1)};$$

and also
$$a(\tan\theta - \tan\beta) = BD$$
$$= \tfrac{1}{2} g t^2 \quad\text{..................(2)}.$$

Eliminating t between (1) and (2) we get

$$1 + \tan^2\theta = \frac{2V^2}{ga}(\tan\theta - \tan\beta),$$

$$(ga\tan\theta - V^2)^2 = V^4 - 2gaV^2\tan\beta - g^2a^2$$
$$= (V^2 - ga\tan\beta)^2 - g^2a^2\sec^2\beta \ldots (3).$$

This equation shews that the least value of V is given by the equation

$$V^2 - ga\tan\beta = ga\sec\beta,$$

whence
$$V^2 = ga\frac{1+\sin\beta}{\cos\beta} = g(b+c),$$

$$V = \{g(b+c)\}^{\frac{1}{2}}.$$

Putting this value of V in the equation (3), we see that

$$\tan\theta = \frac{V^2}{ga} = \frac{b+c}{a} = \frac{1+\sin\beta}{\cos\beta} = \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right),$$

whence
$$\theta = \frac{1}{4}(\pi + 2\beta).$$

(6) Prove that the components of the velocities, at the extremities of any chord of the path of a projectile, at right angles to the chord, are equal.

(7) Three heavy particles are simultaneously projected from the same point and in the same vertical plane; find the relation between the velocities and directions of projection, in order that the three particles may always lie in a straight line.

If V, V', V'', be the velocities of projection, and α, α', α'', the angles between the directions of projection and any assigned straight line, the required relation is

$$\frac{\sin(\alpha'-\alpha'')}{V}+\frac{\sin(\alpha''-\alpha)}{V'}+\frac{\sin(\alpha-\alpha')}{V''}=0.$$

(8) If a ball be projected from a point in an inclined plane, in a direction such that the range on the plane is the greatest possible: prove that the direction of motion, on striking the plane, is perpendicular to the direction of projection. ✓

(9) If a body be projected, at right angles to an inclined plane, with a velocity which would be acquired in falling freely through a space equal to $\frac{3}{8}$ths of the range on the plane, find the inclination of the plane. ✓

The required inclination is equal to $\frac{\pi}{6}$.

(10) A particle begins to slide from rest down an inclined plane AB: at the same instant another particle is projected from A: find the condition that the particles may meet, and when and where this occurs. ✓

The second particle must be projected at right angles to the plane. If α be the inclination of the plane and V the velocity of projection, the time before the meeting takes place will be equal to

$$\frac{2V}{g\cos\alpha},$$

and the distance of the point of meeting from A will be equal to

$$\frac{2V^2\sin\alpha}{g\cos^2\alpha}.$$

(11) A given inclined plane passes through the point of projection of a projectile, which eventually strikes the plane at right angles: find the range of the projectile on the inclined plane, the velocity of projection being given.

If α be the inclination of the plane and V the velocity of projection, the required range is equal to

$$\frac{2V^2}{g} \cdot \frac{\sin\alpha}{1+3\sin^2\alpha}.$$

(12) Shew that the two instants at which a bomb has got a certain angular elevation, when seen from one point in the plane of its motion, are equidistant from the two instants at which it has got the same elevation, when seen from another point in the same plane.

CHAPTER III.

CONSTRAINED MOTION.

SECT. 1. *Direct motion on inclined planes.*

(1) IF two circles, the planes of which are vertical, touch each other internally at their highest or lowest points, and if any chord be drawn within the larger circle, terminating respectively in its highest or lowest point, to prove that the time of descent down that portion of the chord, which is exterior to the smaller circle, is invariable.

Let AB, fig. (109) or fig. (110), be the diameter of the larger circle, AP any chord terminating in A, PQ the portion of AP which is exterior to the smaller circle.

Let $\angle PAB = \theta$, and let D, d, denote the diameters of the larger and smaller circles respectively.

Then the square of the time down PQ is equal to

$$\frac{2PQ}{g\cos\theta} = \frac{2}{g}\frac{AP - AQ}{\cos\theta} = \frac{2}{g}(D - d),$$

and therefore the time is independent of θ.

(2) AP, PB, are chords of a circle, AB, a diameter of the circle, being vertical: particles, starting simultaneously at A, P, fall down AP, PB, respectively: to prove that the least distance between them is equal to the distance of P from AB.

Let $AP = a$, $\angle PAB = \alpha$. Let H, K, be the positions of the particles after any time t: let $HK = d$. Then

$$\begin{aligned} d^2 &= HP^2 + PK^2 \\ &= (a - \tfrac{1}{2}gt^2\cos\alpha)^2 + (\tfrac{1}{2}gt^2\sin\alpha)^2 \\ &= a^2 - agt^2\cos\alpha + \tfrac{1}{4}g^2t^4, \end{aligned}$$

$$(\tfrac{1}{2}gt^2 - a\cos\alpha)^2 = d^2 - a^2\sin^2\alpha :$$

this result shews that $a \sin \alpha$, that is, the distance of P from AB, is the least distance between the particles.

(3) Two circles lie in the same plane, the lowest point of one being in contact with the highest point of the other: to prove that the time of descent from any point of the former to a point in the latter, along a straight line joining these points and passing through the point of contact, is constant.

Let x, y, be the two chords successively described by the descending body, α being the inclination of each to the line joining the centres of the circles. Let r, s, be the radii of the circles, β the inclination of the plane of the circles to the horizon.

The force of gravity g may be decomposed into $g \sin \beta$, parallel to the line of the centres, and $g \cos \beta$, at right angles to the plane of the circles: the latter is ineffective in respect to the body's motion. The effective part of $g \sin \beta$ is its component along the line of motion, which is equal to $g \sin \beta \cos \alpha$: hence, t denoting the time of descent through $x + y$,

$$x + y = \tfrac{1}{2} g t^2 \sin \beta \cos \alpha :$$

but
$$x = 2r \cos \alpha, \quad y = 2s \cos \alpha ;$$

hence
$$2 (r + s) = \tfrac{1}{2} g t^2 \sin \beta, \quad t^2 = \frac{4(r + s)}{g \sin \beta},$$

a result which shews that t is constant.

(4) Any number of smooth fixed straight rods, not in the same plane, pass through a given point: a heavy particle slides down each rod, the particles starting simultaneously from the given point. If the rods be so situated that the particles are, at one instant of their motion, in the same plane, to prove that they will be so throughout their motion, and that a circle can be described passing through them.

If, at one instant of the motion, the particles are in the same plane, it follows that, f being the component of gravity along any rod, the component of f at right angles to the plane must be the same for all the rods: hence the particles will always be in a plane moving parallel to itself.

Again, since any number of particles, descending from a given point at the same instant along fixed straight lines in a vertical

plane, are always in the circumference of a vertical circle, the highest point of which is the given point, it is evident that, in the case of the present problem, they will always be in the surface of a sphere, the highest point of which is the given point. Hence, if the particles are at any instant all in one plane, they must be all in a circle on the sphere.

(5) Particles slide from a fixed point down rough planes to points in the surface of a cone, the axis of which, passing in direction through the point, is vertical, its vertex being upwards: to prove that, if the vertical angle of the cone be equal to $2\tan^{-1}\frac{1}{\mu}$, where μ is the coefficient of friction, the particles will all have the same velocity on arriving at the cone.

Let A, fig. (111), be the fixed point, V the vertex of the cone, VO its axis, and AB one of the inclined planes.

Let $AV=c$, $AB=l$, $\angle BAV=\theta$, $\angle BVO=\omega$, $m=$ the mass of any one of the particles, $R=$ its normal pressure on the plane. Then the accelerating force on a particle down AB is equal to

$$g\cos\theta-\frac{\mu R}{m}$$

$$=g\cos\theta-\mu g\sin\theta$$

$$=g(\cos\theta-\cot\omega\,.\sin\theta)$$

$$=g\frac{\sin(\omega-\theta)}{\sin\omega}=g\frac{c}{l}.$$

Hence the square of the velocity acquired down AB is equal to $2gc$, a constant quantity.

(6) What is the inclination of an inclined plane, such that the accelerating force on a particle sliding down it may be 16·1 feet, the unit of time being a second?

The required inclination is 30°.

(7) A right-angled triangle has its hypotenuse vertical: if three bodies slide down the three sides, prove that the velocities acquired will be proportional to the sides.

(8) One end of a smooth inclined plane is a foot higher than the other, the length of the plane being ten feet: find the time in which a particle will slide down it, and the velocity acquired.

The required time and velocity are approximately 2·49 seconds and 8·02 feet.

(9) Two particles are allowed to slide down an inclined plane from the same point, with an interval of one second between the times of starting: compare their distances from each other at the ends of 1″, 2″, 3″, ... subsequently to the commencement of motion.

The distances between the particles, after 1″, 2″, 3″, ..., are as 1, 3, 5, 7, ...

(10) The two sides of a right-angled triangle are respectively vertical and horizontal: determine the ratio between them, in order that the whole time in which a particle would traverse them, falling down the former and moving along the latter with the velocity acquired in the fall, may be equal to the time of descent down the hypotenuse.

The length of the vertical must be to that of the horizontal side as 3 to 4.

(11) AB is a vertical diameter of a circle: through A, the highest point, any chord AC is drawn, and, through C, a tangent meeting the tangent at B in the point T: shew that the time of a particle's sliding down CT varies inversely as AC.

(12) A is the highest point of a vertical circle, of which O is the centre, B being an extremity of the horizontal diameter: a straight line is drawn touching the circle in P, and cutting OA, OB, produced, in C, D, respectively: prove that, if a particle slide from rest down CD, the times of moving from C to P and from P to D are as ON to NB, where N is the point in which a vertical line through P meets OB.

(13) A tangent at any point P of a circle meets the tangents at the extremities of a vertical diameter AB in C, D, respectively:

if t, t', T, be the times of sliding from rest down CP, PD, CD, respectively, prove that

$$t : t' :: \text{the chord } AP : \text{the chord } BP,$$

and that T varies as tt'.

(14) A is the highest point of a vertical circle, and AB any chord: a circle is described on AB as diameter, and a tangent, drawn at B to the former circle, meets the latter in C: prove that the time of descent down CB is constant.

(15) A body slides from rest down a smooth sloping roof, and then falls to the ground: find the point where it reaches the ground.

Let c = the length of the slope, α = its inclination to the horizon, h = the height of the lowest point of the slope from the ground; then the distance of the point, where the body reaches the ground, from the foot of the wall, is equal to

$$2\cos\alpha \,.\, (c\sin\alpha)^{\frac{1}{2}} \,.\, \{(c\sin^3\alpha + h)^{\frac{1}{2}} - (c\sin^3\alpha)^{\frac{1}{2}}\}.$$

(16) Two equal inclined planes are placed back to back, and a ball, projected up one, flies over the top, and comes to the ground just at the foot of the other: find the velocity of projection, α being the inclination of each plane and h their common altitude.

The required velocity is equal to

$$\tfrac{1}{2}\,(gh)^{\frac{1}{2}} \,.\, (8 + \operatorname{cosec}^2\alpha)^{\frac{1}{2}}.$$

(17) A body is projected from a point A, with the velocity acquired by falling down a height a, up an inclined plane of which the base and height are each equal to b, and, after quitting the plane, strikes the horizontal plane AB in the point B: find AB.

$$AB \text{ is equal to } a + (a^2 - b^2)^{\frac{1}{2}}.$$

(18) A particle slides down a smooth inclined plane: determine the point in which the plane is cut by the directrix of the path described by the particle after leaving the plane.

The directrix intersects the plane at the point where the particle began its motion.

(19) Two equal particles begin, at the same instant, to descend from rest along the chords AP, PB, of a semicircle, the diameter AB of which is vertical: shew that their centre of gravity will descend vertically.

(20) Two equal particles are projected, with equal velocities, up two inclined planes, from their point of intersection, the planes being at right angles to each other: prove that the centre of gravity of the two particles will describe a parabola.

(21) A heavy body is projected up a rough plane, inclined to the horizon at an angle of 60°, with the velocity which it would have acquired in falling freely through a space of 12 feet, and just reaches the top of the plane: find the altitude of the plane, its roughness being such that, if it were inclined to the horizon at an angle of 30°, the body would be on the point of sliding.

The required height is nine feet.

(22) A ring slides down a straight rod, whilst the rod is carried uniformly, in one plane, at a given angle to the horizon: find the path described by the ring.

The path is a parabola.

SECT. 2. *Lines of quickest descent.*

(1) A given point and a given straight line are in the same vertical plane: to determine the straight line of quickest descent from the given point to the given line.

Let P, fig. (112), be the given point, and AB the given straight line: from P draw PX horizontally to meet AB in X: take XY, along XB, equal to XP: join PY: then PY will be the straight line of shortest descent from P to AB.

For, draw PC, YC, at right angles to PX, XY, respectively, to intersect in C. Then the angles XPY, XYP, of the isosceles triangle PXY, are equal, and therefore their respective comple-

ments CPY, CYP, are also equal: hence CP is equal to CY, and therefore a circle described about C, with radius CP or CY, will pass through both P and Y, and, the angles CPX, CYX, being right angles, will touch both PX and BX. Also, the horizontal line PX being a tangent at P, P must be the highest point of the circle.

Hence the time down the chord PY is equal to that down any other chord drawn from P and therefore less than that down any other such chord produced, that is, than the time down any other straight line from P to AB.

(2) A given circle and a given point without it are in the same vertical plane: to determine the straight line of quickest descent from the point to the circle.

Let C, fig. (113), be the centre of the given circle and P the given point. Draw CQ, the vertical downward radius of the circle, and join PQ, intersecting the circumference of the circle in the point R. Then PR is the required straight line of quickest descent.

For, join CR and produce it to meet PO, drawn vertically downwards, in O. Then, in the isosceles triangle QCR,

$$\angle\, CQR = \angle\, CRQ:$$

but, PO, CQ, being both vertical lines, $\angle\, CQR = \angle\, OPR$: also $\angle\, CRQ = \angle\, ORP$: hence $\angle\, OPR = \angle\, ORP$, and therefore $OP = OR$. Hence a circle described about O as centre, with OP as radius, will touch the given circle in R, because CRO is a common normal to the two circles. Also P is the highest point of the circle O.

Hence the time down PR is equal to that down any other chord of the circle O, drawn from P, and is therefore less than the time down any such chord produced, and accordingly than the time down any other straight line drawn from P to the circumference of the circle C.

(3) A given circle and a given point are in the same vertical plane, the point being within the circle: to determine the straight line of quickest descent from the circle to the point.

Let C, fig. (114), be the centre of the given circle, and P the given point. Draw CQ, the vertical downward radius of the circle: join Q, P, and produce QP to intersect the circumference of the circle in R. Then RP is the straight line of quickest descent from the given circle to the given point.

For, join CR, and draw PO, vertically, to meet CR in O. Then, in the isosceles triangle QCR, $\angle\, CQR = \angle\, CRQ = \angle\, ORP$: but, CQ, OP, being parallel to each other, $\angle\, CQR = \angle\, OPR$: hence, $\angle\, OPR = \angle\, ORP$: hence $OR = OP$. Hence if, with centre O and radius OP, a circle be described, it will pass through R: also, OR being a common normal to both circles, the circles will touch each other at R. Hence, P being the lowest point of the circle O, the time down RP will be equal to that down any other chord terminating at P, and will therefore be less than that down any other straight line drawn from the outer circle to P.

(4) A given straight line and a given circle, to which the straight line is exterior, are in the same vertical plane: to determine the straight line of quickest descent from the line to the circle.

Let AB, fig. (115), be the given straight line. Through Q, the lowest point of the circle, draw the horizontal line QX to cut AB in X: along XB measure XY equal to XQ: join YQ: let R be the intersection of YQ and the given circle. Then YR is the straight line of quickest descent from the given line to the given circle.

For, from Q, Y, draw QO, YO, at right angles to XQ, XY, respectively, to meet in O: then a circle, described about O as a centre, with radius OQ, will touch XQ, XY, in Q, Y, respectively.

Now, by Prob. (2), YR is the straight line of quickest descent from the point Y to the given circle: also, if Y' be any other point in AB, and $Y'Q$ be joined, cutting the two circles in Z, R', $Y'R'$ is the straight line of quickest descent from Y' to the given circle. But by Prob. (1), Sect. (1), the time down YR is equal to that down ZR', and therefore less than that

down $Y'R'$. Hence YR is the straight line of quickest descent from the given line to the given circle.

(5) To determine the straight line of quickest descent from one given circle to another given circle, the two circles being in the same vertical plane, and the latter circle being within the former.

Let P, Q, fig. (116), be the lowest points of the outer and inner circles respectively: join PQ and produce it to cut the inner and outer circles in R and S respectively. Then SR is the straight line of quickest descent from the outer to the inner circle.

For, let C be the centre of the inner circle: join RC and produce it to meet PO, drawn vertically from P, in the point O. Join QC. Then, since $QC = RC$, and since PO is parallel to QC, we see that $PO = RO$. If therefore, with O as centre and OP as radius, a circle be described, it will pass through R: it will also touch the two given circles at P, R, respectively.

Now, by Prob. (2), SR is the straight line of quickest descent from the point S to the inner of the two given circles. Take any other point S' in the outer circle: join $S'P$, cutting the inner of the two given circles in R' and the subsidiary circle in Z: then, by Prob. (2), $S'R'$ is the straight line of quickest descent from S' to the inner of the two given circles. But, by Prob. (1), Sect. 1, the time down SR is equal to that down $S'Z$: hence the time down SR is less than that down $S'R'$. Hence, the line SR is the straight line of quickest descent from the outer to the inner of the two given circles.

(6) To find a point at a given distance from the centre of a vertical circle, such that the time of falling from it to the centre is less than the time of falling to any point in the circumference except one, and equal to the time of falling to this point.

Let C, fig. (117), be the centre of the vertical circle; let $r =$ the radius of the circle, and $a =$ the given distance of the required point from C.

Draw vertically $CA = \frac{1}{2}r$: about A, C, as centres, with radii $\frac{1}{2}r$, a, respectively, describe circles cutting each other in P: P is the required point.

For, join AP, CP: draw CO, PO, parallel respectively to AP, AC: produce CO to Q, making $OQ = CO$, Q being accordingly a point in the proposed vertical circle: join PQ.

Since $ACOP$ is a parallelogram,

$$OP = CA = \tfrac{1}{2}r, \quad OC = AP = \tfrac{1}{2}r:$$

hence $OC = OP = OQ$, and therefore a circle, of which P is the highest point and O the centre, can be drawn through the three points C, P, Q, CQ being one of its diameters. Since CQ is a diameter of the circle CPQ and a radius of the proposed vertical circle, they must touch each other at Q.

Since, then, P is the highest point of the circle CPQ, the time down the chord PC is equal to that down any other chord drawn from P, and therefore to that down PQ, but less than the time down any chord, drawn from P, produced, and therefore than the time to any other point in the circumference of the proposed vertical circle except Q.

(7) A given circle and a given point are in the same vertical plane, the point being within the circle: determine the straight line of quickest descent from the point to the circle.

The required straight line is the distance between the given point and the lower end of that chord of the circle which passes through the given point and terminates in the highest point of the circle.

(8) A given point and a given straight line are in the same vertical plane: determine the straight line of quickest descent from the given line to the given point.

From the given point P draw PX horizontally to meet the given line in X: draw upwards along the given line a length XY equal to PX: the straight line joining P and Y is the required straight line.

(9) AC is a given horizontal line, and AB a line elevated above it at an angle α: prove that the least time in which it is

possible for a particle to descend from AB to C, along a straight line, is equal to

$$2\left(\frac{AC}{g}\tan\frac{\alpha}{2}\right)^{\frac{1}{2}}.$$

(10) A given circle and a given point are in the same vertical plane, the point being exterior to the circle: determine the straight line of quickest descent from the circle to the point.

The required straight line is the distance between the given point and the lower end of that chord of the circle which terminates in the highest point of the circle and passes, when produced, through the given point.

(11) A given circle and a given exterior straight line are in the same vertical plane: determine the straight line of quickest descent from the circle to the line.

At the highest point Q of the circle draw a tangent to intersect the straight line in X: measure off downwards along the straight line a length XY equal to XQ: join YQ, cutting the circle in R. Then RY is the straight line of quickest descent from the given circle to the given line.

(12) Determine the straight line of quickest descent from one given circle to another given circle, the two circles being in the same vertical plane and exterior to each other.

Join the highest point of the former circle with the lowest point of the latter: the portion of this line, which lies without both circles, is the required line of quickest descent.

(13) One given circle lies within another given circle, both circles being in the same vertical plane: find the straight line of quickest descent from the inner to the outer circle.

Draw a straight line to intersect the two circles in their highest points: the distance between the second intersections of the two circles by this straight line is the required straight line of quickest descent.

(14) Prove that all points, which lie in the plane of a given vertical circle, and from which the time of quickest descent to

the circle is the same, lie in the circumferences of two circles. Mackenzie and Walton: *Solutions of the Cambridge Problems of* 1854, p. 124.

(15) Tangents are drawn to a vertical circle; find the locus of points in them, from which particles would descend in straight lines to the centre in the shortest time.

The required locus is an indefinite tangent at the highest point of the circle.

SECT. 3. *Lines of slowest descent.*

(1) To determine the straight line of slowest descent from a given point to a given circle, the point being without the circle and both being in the same vertical plane, and the highest point of the circle being lower than the given point.

Let P (fig. 118) be the given point, Q the highest point of the given circle: join PQ, and produce it to cut the circle in R. Then PR is the straight line of slowest descent from the point to the circle. For, let C be the centre of the circle. Join QC, RC, and produce RC to meet PO, drawn vertically from P, in the point O. Then, PO being parallel to QC, and QC being equal to RC, PO must be equal to RO. Hence a circle described about O as a centre, with OP as radius, will pass through R: it will also touch the given circle at R.

Take any point R' in the given circle, not coinciding with R: join PR', and produce it to cut the subsidiary circle in Z.

Then, P being the highest point of the subsidiary circle, the time down PR is equal to that down PZ and therefore greater than that down PR'. Hence PR is the required straight line of slowest descent.

(2) Determine the line of slowest descent from a given circle to a given point without it, the point and circle being in the same vertical plane, and the point being lower than the lowest point of the circle.

From the given point draw an indefinite straight line, cutting the circle in its lowest point: then the distance between the

given point and the second intersection of the indefinite line and the circle is the required line of slowest descent.

(3) Find the line of slowest descent from one given circle to another given circle, both circles being in the same vertical plane and each being exterior to the other; the highest point of the latter circle being lower than the lowest of the former.

Produce the line, which joins the lowest point of the former circle and the highest point of the latter, to meet both circles again: then the distance between the second intersections is the required line of slowest descent.

SECT. 4. *Oblique motion on inclined planes.*

(1) A body is projected from a given point in a horizontal direction with a given velocity, and moves upon an inclined plane passing through the point: if the inclination of the plane vary, find the locus of the directrix of the parabola which the body describes.

If V be the velocity of projection, the locus of the directrix is a horizontal plane at an elevation above the point of projection equal to $\dfrac{V^2}{2g}$.

(2) A particle is projected horizontally, with a given velocity, along a plane inclined at a given angle to the horizon: find the velocity with which a body must be projected horizontally in free space, so that the parabolas described may be equal.

If $u =$ the given velocity and $\alpha =$ the given angle of inclination, the required velocity is equal to

$$\frac{u}{(\sin\alpha)^{\frac{1}{2}}}.$$

SECT. 5. *Motion on curves.*

(1) One end of a fine string is attached to an angular point B of a fixed regular polygon $ABCD$... on a fixed horizontal plane, its length being equal to the perimeter: a particle P, fixed to the other end of the string, which is stretched in the direction

AB, is projected in the plane of the polygon, perpendicularly to the string, with a given velocity, so that the string comes into contact with BC, CD,...successively: to determine after what time the string will coincide with the perimeter of the polygon.

Let V= the velocity of projection, which will be the velocity of the particle throughout the motion: let a= the length of each side of the polygon, n= the number of the sides. Let ω_1, ω_2, ω_3,.... be the angular velocities of the successively free portions BP, CP, DP,.... of the string. Then,

$$na, \quad (n-1)\,a, \quad (n-2)\,a, \ldots a,$$

being the radii of the successive circular arcs described by P,

$$\omega_1 = \frac{V}{na}, \quad \omega_2 = \frac{V}{(n-1)\,a}, \quad \omega_3 = \frac{V}{(n-2)\,a}, \ldots. \omega_n = \frac{V}{a}.$$

But, t_1, t_2, t_3, ... t_n, being the times of the description of the successive circular arcs, we have, $\frac{2\pi}{n}$ being the angle subtended by each of these arcs at their respective centres,

$$t_1 = \frac{2\pi}{n\omega_1} = \frac{2\pi a}{V},$$

$$t_2 = \frac{2\pi}{n\omega_2} = \frac{2\pi a}{V} \cdot \frac{n-1}{n},$$

$$t_3 = \frac{2\pi}{n\omega_3} = \frac{2\pi a}{V} \cdot \frac{n-2}{n},$$

$$t_n = \frac{2\pi}{n\omega_n} = \frac{2\pi a}{V} \cdot \frac{1}{n}.$$

Hence the whole time occupied by the string in winding itself about the polygon is equal to

$$+ t_2 + t_3 \ldots + t_n = \frac{2\pi a}{nV}\{n + (n-1) + (n-2) + (n-3) + \ldots + 1\}$$

$$= \frac{\pi a}{V}(n+1).$$

Let l = the length of the string. Then the whole time is equal to

$$\frac{\pi l}{V} \cdot \frac{n+1}{n}.$$

Cor. Suppose the number of the sides to be infinite, when the polygon becomes a circle: then the whole time is equal to

$$\frac{\pi l}{V}.$$

(2) A smooth tube, of uniform bore, is bent into the form of a circular arc, greater than a semicircle, and placed in a vertical plane with its open ends upwards and in the same horizontal line: find the velocity with which a ball, that fits the tube, must be projected along the interior from the lowest point, in order that it may pass out at one end and re-enter at the other.

If r = the radius of the circle, h = the depth of the centre of the circle below the horizontal line through the two ends of the tube, and V = the required velocity; then

$$V^2 = \frac{g}{h}(r^2 + 2hr + 2h^2).$$

(3) A particle slides from rest down a narrow smooth tube in the form of the thread of a screw, the axis of which is vertical: find the time in which it will make a complete revolution about the axis.

If a = the radius of the cylinder on which the helix is described, and α = the angle which the thread makes with a generating line of the cylinder, the required time is equal to

$$\left(\frac{8\pi a}{g \sin 2\alpha}\right)^{\frac{1}{2}}.$$

(4) Shew that, if a particle descend down a cycloid, of which the axis is vertical and the vertex downwards, from an extremity of the base, the velocity at any point will be proportional to the radius of curvature at the point.

SECT. 6. *Pendulums.*

(1) If a clock pendulum lose 5″ a day, to determine the alteration which must be made in its length.

Let l be the length of a seconds pendulum, $l+\alpha$ of the pendulum under consideration. Then the number of seconds in the time of the oscillation of the latter pendulum is equal to

$$\pi\left(\frac{l+\alpha}{g}\right)^{\frac{1}{2}}=\left(\frac{l+\alpha}{l}\right)^{\frac{1}{2}}:$$

hence the number of oscillations which it performs in 24 hours is equal to

$$24\times 60\times 60\times\left(\frac{l}{l+\alpha}\right)^{\frac{1}{2}}$$

$$=24\times 60\times 60\times\left(1-\frac{\alpha}{2l}\right),\text{ nearly,}$$

and consequently

$$5=24\times 60\times 60-24\times 60\times 60\times\left(1-\frac{\alpha}{2l}\right)$$

$$=24\times 60\times 60\times\frac{\alpha}{2l},$$

and therefore

$$\frac{\alpha}{l}=\frac{1}{8640}:$$

thus we see that the pendulum must be diminished by very nearly the $(8640)^{\text{th}}$ part of its length.

(2) A seconds pendulum was too long on a given day by a quantity α; it was then over-corrected so as to be too short by α during the next day: to prove that, l being the length of the seconds pendulum, the number of minutes gained in the two days was

$$1080\,\frac{\alpha^2}{l^2},\text{ nearly.}$$

The time of each oscillation in seconds during the first day was equal to

$$\pi\left(\frac{l+\alpha}{g}\right)^{\frac{1}{2}},$$

and therefore the number of apparent seconds was equal to

$$\frac{24 \times 60 \times 60}{\pi \left(\frac{l+\alpha}{g}\right)^{\frac{1}{2}}},$$

and the number of apparent minutes to

$$\frac{24 \times 60}{\pi \left(\frac{l}{g}\right)^{\frac{1}{2}}} \cdot \left(1 + \frac{\alpha}{l}\right)^{-\frac{1}{2}}$$

$$= 24 \times 60 \times \left(1 + \frac{\alpha}{l}\right)^{-\frac{1}{2}}$$

$$= 24 \times 60 \times \left(1 - \frac{\alpha}{2l} + \frac{3\alpha^2}{8l^2}\right), \text{ nearly.}$$

Similarly, putting $-\alpha$ for α, the number of apparent minutes on the second day was equal to

$$24 \times 60 \times \left(1 + \frac{\alpha}{2l} + \frac{3\alpha^2}{8l^2}\right), \text{ nearly.}$$

Hence the number of minutes gained in the two days was equal to

$$24 \times 60 \times \left(2 + \frac{3\alpha^2}{4l^2}\right) - 2 \times 24 \times 60$$

$$= 1080 \frac{\alpha^2}{l^2}.$$

(3) A seconds pendulum, carried up to the top of a mountain, is found to lose there 43″.2 a day: to find the height of the mountain, supposing the radius of the Earth to be 4000 miles.

Let l be the length of the pendulum, g the force of gravity at the foot and g' at the summit of the mountain. Let t denote the time of an oscillation at the summit, and x the height of the mountain in miles.

Then $$t = \pi \sqrt{\frac{l}{g'}}:$$

but, since the force of gravity, in ascending above the Earth's surface, varies inversely as the square of the distance from the centre,

$$g' = g\,\frac{(4000)^2}{(4000+x)^2}:$$

hence
$$t = \pi\sqrt{\frac{l}{g}}\cdot\frac{4000+x}{4000}$$

$$= 1 + \frac{x}{4000}.$$

Again, by the hypothesis,

$$\frac{24 \times 60 \times 60}{t} = 24 \times 60 \times 60 - 43.2,$$

$$t = 1 + \frac{43.2}{24 \times 60 \times 60}, \text{ nearly.}$$

Hence
$$x = \frac{43.2 \times 4000}{24 \times 60 \times 60} = 2 \text{ miles.}$$

(4) Find the time of oscillation of a pendulum, 20 feet long.

The required time is, approximately, two seconds and a half.

(5) Find the length of a pendulum which oscillates in half minutes.

The required length is, approximately, 978 yards.

(6) A body, dropped from the top of a wall, falls to the ground, while a pendulum, 6 inches long, makes 5 oscillations: find the height of the wall.

The required height is $\frac{25}{4}\pi^2$ feet.

(7) A seconds pendulum is lengthened one hundredth of an inch: find how many seconds it will lose daily.

About eleven seconds will be lost daily.

(8) If a simple pendulum, $39\frac{1}{8}$ feet long, oscillates in 1″; find the length of one which loses a minute in an hour.

The required length is $40\frac{40}{59}$ feet.

(9) If 39.1386 inches be the length of a seconds pendulum; what will be the length of one which vibrates 40 times in a minute?

The required length is 88.06185 inches.

(10) If the length of a seconds pendulum be 39.1393 inches, find the value of g to three places of decimals.

The value of g is 32.190 feet.

(11) A pendulum, which would oscillate seconds at the equator, would, if carried to the pole, gain five minutes a day: compare the polar and equatoreal gravity.

The gravity at the equator is to the gravity at the pole as 144 to 145, approximately.

(12) A pendulum, which oscillates seconds at one place, is carried to a place where it gains two minutes a day: compare the force of gravity at the latter place with that at the former.

If g denote the force of gravity at the former and g' at the latter place, then, approximately,

$$g' : g :: 361 : 360.$$

(13) A pendulum is found to make 640 vibrations at the equator in the same time in which it makes 641 at Greenwich; if a string hanging vertically can just sustain 80 pounds at Greenwich, how many such pounds can the same string sustain at the equator?

The required number of such pounds is nearly $80\frac{1}{4}$.

(14) Two pendulums, the lengths of which are l and l', are oscillating at different points on the Earth's surface: the number of vibrations which they respectively make in the same time are in the ratio $m : m'$; find the ratio between the forces of gravity at the two places.

The required ratio is equal to $\frac{lm^2}{l'm'^2}$.

(15) A seconds pendulum is carried to the top of a mountain, of which the height is one mile; find the number of seconds which it will lose daily, gravity being supposed to vary inversely

as the square of the distance from the centre of the Earth, and the radius of the Earth to be four thousand miles.

The number of seconds lost daily is, approximately, 21.6.

(16) A seconds pendulum is carried to the top of a mountain 3000 feet high; assuming that the force of gravity varies inversely as the square of the distance from the Earth's centre, and that the Earth's radius is 4000 miles, find the number of oscillations lost in a day. Also determine how much the pendulum must be shortened in order that it may oscillate seconds on the mountain?

The number of seconds lost in a day is nearly 12, and, in order to correct this, the pendulum must be shortened by about the 3520th part of its length.

(17) A balloon was found to be sailing steadily with the wind, at an invariable elevation above the Earth: a seconds pendulum, suspended in the car, was observed in 50 minutes to make 3003 oscillations: determine the height of the balloon, the Earth's radius being 4000 miles nearly.

The height of the balloon was about four miles.

(18) A seconds pendulum, carried to the summit of a mountain, is found to lose 8 seconds a day: determine approximately the height of the mountain; gravity on the summit of a mountain, of height h, being supposed to be equal to $g\left(1-\frac{5h}{4r}\right)$, where r is the radius of the Earth, 4000 miles nearly.

The altitude is approximately $\frac{16}{27}$ths of a mile.

(19) A seconds pendulum, hanging against the face of a slightly inclined smooth wall, and swinging in its plane, is observed to lose s seconds in t hours; find the inclination of the wall to the vertical.

The circular measure of the inclination of the wall to the vertical is nearly equal to

$$\frac{1}{30}\cdot\left(\frac{s}{t}\right)^{\frac{1}{2}}.$$

SECT. 7. *Illustrations of the third law of motion.*

(1) Weights P and Q are connected by a fine string, of given length, which is hung over a fixed pully: P, starting from the highest point, draws up Q: after some time the string is cut, and then Q rises just to the top: to find the position of P at the instant when the string was cut.

Let c be the length of the string; x, y, the distances of P, Q, respectively, below the pully, at the instant the string was cut, and V their common velocity at the same moment.

Then $$V^2 = 2\,\frac{P-Q}{P+Q} \cdot g \cdot x,$$

and, since y is the altitude due to the velocity V,

$$V^2 = 2gy.$$

Hence $$y = x\,\frac{P-Q}{P+Q}:$$

but $$x + y = c:$$

hence $$x = \frac{c}{2} \cdot \frac{P+Q}{P}.$$

(2) A balloon is descending with a uniformly accelerated velocity, so that a one pound weight, as measured by a spring, appears to weigh only 15 ounces: to find the number of feet through which the balloon will have travelled vertically downwards in the two minutes after it begins to descend, considering the force of gravity constant.

The acceleration of the balloon, being the same as that of the weight, must be equal to

$$\frac{1 \text{ pound} - 15 \text{ ounces}}{1 \text{ pound}} \times 32{\cdot}2 \text{ feet}$$

$$= \frac{32{\cdot}2}{16} \text{ feet.}$$

Hence the required space, expressed in feet, is equal to

$$\frac{1}{2} \times \frac{32 \cdot 2}{16} \times (120)^2$$

$$= 14490, \text{ approximately.}$$

(3) Two bodies, the weights of which are W and W', hang from the extremities of a cord passing over a smooth peg; if, at the end of each second from the beginning of motion, W be suddenly diminished and W' suddenly increased, so as not to experience any impulse, by $\frac{1}{n}$th of their original difference; shew that, W being the greater weight, their velocity will be zero at the end of $n+1$ seconds.

Since the acceleration of the weights varies as the ratio which their difference bears to their sum, it follows that, f denoting their acceleration during the first second, the accelerations during the 2nd, 3rd, 4th, ... $(n+1)^{\text{th}}$ seconds, will be

$$f\left(1-\frac{2}{n}\right), \quad f\left(1-\frac{4}{n}\right), \quad f\left(1-\frac{6}{n}\right), \ldots \quad f\left(1-\frac{2n}{n}\right),$$

and therefore their velocity at the end of the $(n+1)^{\text{th}}$ second will be equal to

$$f\left\{1+\left(1-\frac{2}{n}\right)+\left(1-\frac{4}{n}\right)+\left(1-\frac{6}{n}\right)+\ldots+\left(1-\frac{2n}{n}\right)\right\}$$

$$=f\left\{n+1-\frac{2}{n}\cdot\frac{n+1}{2}\cdot n\right\}=0.$$

(4) A string, passing over a smooth pully, supports, at one end, a scale-pan of weight W, which contains a weight $5W$, and, at the other end, a weight $3W$: to find the tension of the string and the pressure of the weight on the pan.

Let R = the mutual pressure between the pan and the weight $5W$, and T = the tension of the string.

Then, the downward acceleration of the pan is equal to

$$\frac{W+R-T}{W}g \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the downward acceleration of the weight $5W$ to

$$\frac{5W-R}{5W}g \text{ ....(2)},$$

and the upward acceleration of the weight $3W$, to

$$\frac{T-3W}{3W}g \text{ ....(3)}.$$

Since these accelerations are all equal, we have, from (1) and (2),

$$5(R-T) = -R,$$

$$6R = 5T \text{ .... (4)}.$$

From (2) and (3),

$$3(5W-R) = 5(T-3W),$$

$$30W = 3R + 5T \text{ .... (5)}.$$

From (4) and (5) we see that

$$R = \frac{10}{3}W, \text{ and } T = 4W.$$

(5) A particle is projected up a rough inclined plane of indefinite length; to compare the velocities at the beginning of the ascent and the end of the descent.

Let α = the inclination of the plane, $\tan\epsilon$ = the coefficient of friction, m = the mass of the particle.

The normal pressure of the particle on the plane is equal to $mg\cos\alpha$: hence the friction is equal to $mg\cos\alpha\tan\epsilon$. Thus, while the particle is ascending, it is acted on by a retarding force equal to

$$g(\sin\alpha + \cos\alpha\tan\epsilon),$$

and, while it is descending, by an accelerating force equal to

$$g(\sin\alpha - \cos\alpha\tan\epsilon).$$

Hence, s being the length of the inclined plane traversed by the particle, and u, v, the velocities at the beginning of the ascent and the end of the descent, respectively,

$$u^2 = 2gs(\sin\alpha + \cos\alpha\tan\epsilon),$$

and $$v^2 = 2gs(\sin\alpha - \cos\alpha\tan\epsilon);$$

and therefore $$\left(\frac{v}{u}\right)^2 = \frac{\sin(\alpha - \epsilon)}{\sin(\alpha + \epsilon)}.$$

(6) Two equal particles, connected by a fine string, are placed upon two given inclined planes which have a common altitude: to determine the acceleration of the centre of gravity of the two particles.

Let AC, BC, fig. (119), be the two inclined planes. Let $\angle BAC = \alpha$, $\angle ABC = \beta$. Produce BC indefinitely to B', and draw CD to bisect the angle ACB'.

The accelerations of the particles, in the directions CA, CB', respectively, are both equal to

$$\frac{1}{2} g (\sin \alpha - \sin \beta) = g \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}:$$

it is evident therefore that their accelerations at right angles to CD are equal and in opposite directions, and that the acceleration of each parallel to CD is equal to

$$g \sin \frac{\alpha - \beta}{2} \cos^2 \frac{\alpha + \beta}{2}.$$

This is therefore, the particles being equal, also the acceleration of their centre of gravity.

(7) Find the weight of a body which, initially at rest, describes in a certain time, under a pressure of six pounds, one-third of the space through which it would in the same time fall freely.

The weight of the body is eighteen pounds.

(8) A force, which can support 50 lbs., acts for one minute on a body the weight of which is 200 lbs.: find the velocity and momentum acquired by the body.

The velocity acquired is 483 feet per second and the momentum is 3000.

(9) Two weights, one of 81 and the other of 80 ounces, are suspended by a fine string over a fixed pully: determine the

space described by each of them from rest in one second, and the velocity acquired.

The required space and velocity are respectively one tenth and one fifth of a foot.

(10) Two weights, each of 7 lbs., hang at rest by means of a string over a fixed pully, the mass of which may be neglected: to one of them an additional weight of 2 lbs. is attached, which is removed after the weight has fallen through 8 feet: find the time occupied in falling through the 8 feet, and also the time in which 8 feet would be described after the removal of the additional weight, supposing 32 feet to be the measure of gravity.

The required times are respectively 2 seconds and 1 second.

(11) A weight of 4 lbs., hanging down vertically, draws another of 6 lbs. up a plane inclined at an angle of 30° to the horizon: find the space moved through in four seconds.

The required space is 25·76 feet.

(12) A body, weighing 10 pounds, is moved along a horizontal plane by a constant force, which generates in the body in a second a velocity of 1 foot per second; find what weight the force would support.

It would support rather less than 5 ounces.

(13) If a weight of ten pounds be placed upon a horizontal plane, which is made to descend with a uniform acceleration of 10 feet per second, determine the pressure on the plane.

The required pressure is a little greater than 6 lbs. 14 oz.

(14) Two bodies, weighing respectively 1 lb. and 2 lbs., are placed upon a horizontal plane, which descends with a uniform acceleration of 1 foot per second: determine the pressure which each exerts on the plane.

The required pressures are, approximately, $15\frac{1}{2}$ oz. and 31 oz.

(15) Two weights are connected by a fine string which passes over a pully: if the weights be 50 and 72 lbs., determine what stationary weight the string must be able to support, that it may just escape breaking during the motion.

It must be able to support a stationary weight of $59\frac{1}{61}$ lbs.

(16) Two equal bodies P and Q, connected by a string, six feet long, are placed on a horizontal table, three feet high: P is gently pushed over the edge, and falls, drawing Q along the table: find when each will fall to the ground.

Time being dated from the commencement of the motion, P and Q will reach the ground respectively in the times

$$2\left(\frac{3}{g}\right)^{\frac{1}{2}} \text{seconds and } \left(\frac{3}{g}\right)^{\frac{1}{2}}.(3+\sqrt{2}) \text{ seconds.}$$

(17) A weight P, descending vertically, draws a weight $2P$ up an inclined plane, by means of a string passing over a pully at the top of the plane: the ascending weight starts from the foot of the plane, and, when it has travelled half-way up the plane, the motion of the descending weight is stopped: find the inclination of the plane, that the ascending weight may just reach the top.

The required angle of inclination is equal to $\sin^{-1}\left(\frac{1}{5}\right)$.

(18) A body, of weight $4P$, is drawn up a plane, of length a, inclined at 30° to the horizon, by a weight $3P$, connected with the former by a string passing over the upper edge of the plane: find the tension of the string and the time before the former weight arrives at the top of the plane.

The tension of the string is equal to $\frac{18}{7}P$, and the required time is equal to $\left(\frac{14a}{g}\right)^{\frac{1}{2}}$.

(19) A chord AB of a circle is vertical, and subtends at the centre an angle $2\cot^{-1}\mu$: shew that the time down any chord AC, drawn in the smaller of the two segments into which AB divides the circle, is constant; AC being rough, and μ being the coefficient of friction.

(20) A heavy body is placed on a smooth inclined plane, which at the same instant begins to descend with a uniform acceleration of 5 feet in a second, the inclination of the plane to the horizon being invariable: determine the motion of the body and its pressure on the plane.

If α be the inclination of the plane to the horizon, and W the weight of the body, the body will descend in a straight line inclined to the vertical at an angle

$$\tan^{-1}\left(\frac{27\cot\alpha}{32+5\cot^2\alpha}\right),$$

its pressure on the plane being equal to $\frac{27}{32} W \sin\alpha$.

(21) A balloon ascends with a uniformly accelerated velocity, so that a weight of one pound produces, on the hand of the aeronaut sustaining it, a downward pressure equal to that which a pound weight, augmented by the hundredth part of an ounce, would produce at the Earth's surface: find the height which the balloon will have attained in ten minutes from the time of starting, not taking into account the variation of the accelerating effect of the Earth's attraction.

The required height, supposing gravity to be represented by 32 feet, will be twelve hundred yards.

(22) Two weights, one of which is double of the other, are placed upon two smooth inclined planes which have a common vertex, the inclination of each being 30°, and are connected by means of a fine string passing over a small pully at the highest point of the planes: prove that, if the length of the string be equal to that of either plane, and the heavier weight start from the highest point, the two weights will reach the ground at the same time, if the string be cut when one sixth of it has passed over the pully.

Shew also that the times of motion, before and after cutting the string, are each equal to the time of a body's falling freely under the action of gravity through a height equal to the length of either plane.

(23) A bucket of water, weighing 32 lbs., is attached to a rope, which passes round a fixed pully, and is pulled by a man: supposing the man to raise the bucket two feet each stroke, which occupies half a second, and the force to be so applied that the velocity is uniformly generated during the first half of the stroke, and uniformly destroyed during the second half, find how

the tension of the rope alters, and express in pounds the greatest tension, 32 feet being assumed as the measure of the force of gravity.

The tension is 64 lbs. during the first half, and zero during the latter half of each stroke.

(24) A string, charged with $m+n+1$ equal weights, fixed at equal intervals along it, is placed within a smooth bent tube, curved at the angle, and just wide enough to admit the weights, the two branches of the tube being rectilinear, one inclined to the horizon and the other vertical: the string, which would rest with m of the weights within the vertical branch, is so placed that the $(m+1)^{\text{th}}$ weight is just within this branch: shew that, if a be the distance between each two adjacent weights, the velocity which the string will have acquired, at the instant the last weight enters the vertical branch, will be $(nag)^{\frac{1}{2}}$.

Campion and Walton's *Solutions of the Cambridge Problems of* 1857.

(25) A weight, descending vertically, draws another weight, n times as heavy as itself, up a rough inclined plane, by means of a fine string passing over a pully fixed at the top of the plane: the ascending weight starts from the foot of the plane, and, when it has travelled a space x, the connecting string is cut: shew that the ascending weight will just reach the top of the plane if

$$x = \frac{a\,(1+n)}{1+\cos\epsilon\,\operatorname{cosec}\,(\alpha+\epsilon)},$$

where α is the inclination of the plane, a its length, and $\tan\epsilon$ the coefficient of friction between the plane and the weight.

(26) If weights W, nW, move on two inclined planes and be connected by a fine string passing over the common vertex of the planes, the angles of inclination of the planes to the horizon being α, β, respectively, prove that their centre of gravity describes a straight line with a uniform acceleration equal to

$$g\cdot\frac{n\sin\beta-\sin\alpha}{(n+1)^2}\cdot\{n^2+2n\cos(\alpha+\beta)+1\}^{\frac{1}{2}}.$$

SECT. 8. *Centrifugal Force.*

(1) Two unequal weights are connected by a string of given length, which passes through a small ring: to find how many times in a second the lighter must revolve, in order that the heavier may be at rest at a given distance from the ring.

Let P, P', fig. (120), be the positions of the two weights, O the ring, θ being the inclination of OP' to the vertical.

Let $OP' = c'$, and let m, m', denote the masses of P, P', respectively.

The forces acting on P' are the tension of the string, which is equal to mg, the weight $m'g$, and the centrifugal force $m'\omega^2c'$ $\sin\theta$, acting horizontally.

For the equilibrium of P' we have, resolving horizontally,

$$m'\omega^2c' \sin\theta = mg \sin\theta,$$

and therefore
$$\omega^2 = \frac{mg}{m'c'}.$$

Hence, n denoting the number of revolutions performed by P' in a second,

$$n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \cdot \left(\frac{mg}{m'c'}\right)^{\frac{1}{2}}.$$

(2) A string will just bear a weight of 16 lbs. without breaking: if a weight of $\frac{1}{2}$ lb. be attached to it, and whirled round in a horizontal circle, the radius of which is two feet, find the number of revolutions the weight must make in a second, in order to break the string.

The required number of revolutions is equal to $\frac{2}{\pi}\sqrt{g}$.

CHAPTER IV.

IMPACT.

(1) A PARTICLE is projected, with a given velocity, from one extremity of a diameter of a horizontal circle, and, after reflection at the curve, passes through the other extremity: to find the elasticity, in order that the time of motion may be n times that of describing the diameter with the velocity of projection.

Let AB, fig. (121), be the diameter, APB the path of the particle, C the centre of the circle.

Join CP: let $r =$ the radius, $\angle PAC = \alpha$, $V =$ the velocity of projection.

Since the component of the velocity, at right angles to CP, is the same before and after impact, and since A, B, are equidistant from the diameter PCP', it follows that the time of movement from A to P is equal to that from P to B: let t denote each of these times.

Since the line AP is described, in the time t, with the velocity V,

$$2r \cos \alpha = V . t \qquad \text{(1)}.$$

Considering the component of the motion from P to B, parallel to PC, we have, since $eV \cos \alpha$ is the normal component of the velocity after impact on the curve at P,

$$r(1 - \cos 2\alpha) = eV \cos \alpha . t,$$

or

$$2r \sin^2 \alpha = eV \cos \alpha . t \qquad \text{(2)}.$$

Again, the time of motion being equal to that of describing AB with the velocity V, we have

$$Vt = nr \text{ ....} (3).$$

From (1), (2), (3), we get

$$2\cos\alpha = n \text{ ....} (4),$$

and $$2\sin^2\alpha = ne\cos\alpha \text{ ....} (5).$$

From (4) and (5), by the elimination of α, we find that

$$e = \frac{4-n^2}{n^2} \text{ ....} (6).$$

From the equation (4), we see that n must be less than 2, and, from (6), that $4-n^2$ must be less than n^2, or n greater than $\sqrt{2}$. Thus, that the problem may be possible, it is necessary that n be less than 2 and greater than $\sqrt{2}$.

(2) An imperfectly elastic ball is projected in a given direction within a fixed horizontal hoop, so as to go on rebounding from the surface of the hoop: to find the limit to which the velocity of the ball will approach; and to shew that it will attain this limit at the end of a finite time.

Let V, V_1, V_2, V_3, ... be the velocities of the successive impacts; α, α_1, α_2, α_3, ... the acute angles between the successive directions of impact and the tangents to the circle at the respective points of impact.

Then, the tangential components of the velocities of impact and rebound being equal at each point of impact, we have

$$V\cos\alpha = V_1\cos\alpha_1 = V_2\cos\alpha_2 = \text{......} = V_n\cos\alpha_n \text{.........} (1).$$

Again, resolving normally the velocities of impact and rebound,

$$V_1\sin\alpha_1 = eV\sin\alpha:$$

but $$V_1\cos\alpha_1 = V\cos\alpha:$$

hence $$\tan\alpha_1 = e\tan\alpha,$$

and similarly $\tan\alpha_2 = e\tan\alpha_1,$

$$\tan\alpha_3 = e\tan\alpha_2,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$\tan\alpha_n = e\tan\alpha_{n-1}.$$

Multiplying these last n equations together and rejecting factors common to both sides, we have

$$\tan\alpha_n = e^n\tan\alpha.$$

When $n = \infty$, we see from this equation that $\tan\alpha_n = 0$, and therefore, by (1), that the terminal velocity of the ball is $V\cos\alpha$.

Again, r denoting the radius of the hoop, the time through the chord between the n^{th} and $(n+1)^{\text{th}}$ impact, is equal to

$$\frac{2r\sin\alpha_n}{V_n} < \frac{2r\sin\alpha_n}{V_n\cos^2\alpha_n} = \frac{2r\tan\alpha_n}{V_n\cos\alpha_n} = \frac{2r\,e^n\tan\alpha}{V\cos\alpha}:$$

hence, the whole time between the first and $(n+1)^{\text{th}}$ impact is less than

$$\frac{2re\tan\alpha}{V\cos\alpha}(1+e+e^2+\dots+e^{n-1}) = \frac{2re\tan\alpha}{V\cos\alpha}\cdot\frac{1-e^n}{1-e},$$

and therefore the ball arrives at its terminal velocity in a time less than

$$\frac{2r}{V}\cdot\frac{\tan\alpha}{\cos\alpha}\cdot\frac{e}{1-e}.$$

(3) A particle, of given elasticity e, is projected along a horizontal plane, from the middle point of one of the sides of an isosceles right-angled triangle, so as, after reflection at the hypotenuse and remaining side, to return to the same point; to prove that the cotangents of the angles of reflection are $e+1$ and $e+2$ respectively.

Let ABC, fig. (122), be the triangle, A being the right angle: let $PQRP$ be the course of the particle: let

$\angle PQB = \theta'$, $\angle RQC = \theta$, $\angle QRC = \phi'$, $\angle PRA = \phi$.

Then $$\tan\theta = e\tan\theta' \quad\text{.......................}\quad (1),$$

and $$\tan\phi = e\tan\phi'$$

$$= e\tan\left(\frac{3\pi}{4} - \theta\right), \text{ by the geometry,}$$

$$= e\frac{1+\cot\theta}{1-\cot\theta} \quad\text{.....................}\quad (2).$$

Again, by the geometry,

$$PQ\sin(\theta+\theta') = PR\sin(\phi+\phi'),$$

$$\frac{BP\sin\frac{\pi}{4}}{\sin\theta'}.\sin(\theta+\theta') = \frac{AP}{\sin\phi}\sin(\phi+\phi')$$

$$= \frac{AP}{\sin\phi}\sin\left(\frac{\pi}{4}+\theta-\phi\right),$$

whence, AP being equal to BP,

$$\cot\theta' + 2\cot\theta = \cot\phi\,(1+\cot\theta) + 1,$$

and therefore, by (1) and (2),

$$(e+2)\cot\theta = \frac{1}{e}(1-\cot\theta) + 1,$$

$$(e^2+2e+1)\cot\theta = e+1,$$

$$\tan\theta = e+1,$$

and therefore, by (2),

$$\tan\phi = e+2.$$

(4) A locomotive is travelling at the rate of 60 miles an hour: shew that, if it were suddenly stopped, the violence of the concussion would be nearly as great as if it fell from a height of 40 yards.

(5) A ball, of given elasticity, impinges against a circular arc: determine the point of impact, in order that it may rebound at right angles to its direction of incidence.

Let A be the point of the arc, the radius of which is parallel to the direction of incidence, and let P be the point of impact: then, r denoting the radius and e the elasticity,

$$AP = r\tan^{-1}(e^{\frac{1}{2}}).$$

(6) ABC is a horizontal triangle; D, E, F, the points where the circle inscribed in it meets the sides BC, CA, AB, respectively: prove that, if a ball, of elasticity e, be projected from D, so as to strike AC in E and then rebound to F, $AE = e \cdot CE$: if the ball return to D, prove that also $AB = e \cdot AC$.

(7) A ball is projected from the middle point of one side of a billiard-table, so as to strike in succession one of the sides adjacent to it, the side opposite to it, and a ball placed in the centre of the table; shew that, if a and b be the lengths of the sides of the table, and e the elasticity of the ball, the inclination of the direction of projection to the side a of the table, from which it is projected, must be

$$\tan^{-1}\left(\frac{b}{a} \cdot \frac{1+2e}{1+e}\right).$$

(8) The tangents of the angles of a triangle ABC are in geometrical progression, $\tan B$ being the mean proportional: a ball is projected in a direction parallel to the side CB, so as to strike the sides AB, BC, successively: shew that, if its course after the first impact be parallel to AC, its course after the second will be parallel to BA; and that, if e be the modulus of elasticity,

$$\sec B = e^{\frac{1}{2}} + e^{-\frac{1}{2}}.$$

(9) A heavy particle, of given elasticity, impinges on a fixed rough plane in a given direction: determine the direction of the motion of the particle after impact, the impulsive friction being proportional to the impulsive pressure between the particle and plane.

If α, α', denote the angles of incidence and reflection, e the elasticity of the particle, and μ the coefficient of friction,

$$\tan \alpha' = \frac{1}{e} \tan \alpha - \frac{\mu}{e}(1+e).$$

(10) Shew that it is possible to project a ball on a smooth billiard-table from a given point in an infinite number of directions, so as, after striking all the sides in order once or oftener, to hit another given point; but that this number is limited, if it have to return to the point from which it was projected.

CHAPTER V.

COLLISION OF SPHERES.

SECT. 1. *Direct Collision.*

(1) Two spheres A and B, moving with equal velocities in opposite directions, collide directly: supposing that, after collision, A is at rest, and that B moves back with the velocity which it had before collision, to compare the masses of A and B.

Let u denote the velocity of either sphere before collision, and let m, m', represent their respective masses: then, the algebraic sum of the momenta being the same after collision as before, we have

$$mu - m'u = m'u,$$

and therefore

$$m : m' :: 2 : 1.$$

(2) A ball A strikes a ball B, which is at rest, directly, and, after collision, their velocities are equal and opposite: to find the mutual elasticity of the balls, supposing the mass of B to be λ times that of A.

Let m be the mass of A, u its velocity before, and v its velocity, in the opposite direction, after collision: the mass of B is accordingly λm, and its velocity v. Let e be the common elasticity of the balls.

Then, the algebraic sum of the momenta being the same before and after collision,

$$mu = -mv + \lambda mv,$$

or

$$u = (\lambda - 1)\, v \quad \text{.......................} \quad (1).$$

Again, by the equation of elasticity, u being the relative velocity before and $2v$ after collision,

$$eu = 2v \text{ ....} (2).$$

From (1) and (2) we see that

$$e = \frac{2}{\lambda - 1}.$$

COR. Since e is less than unity, we must have $2 < \lambda - 1$, or $\lambda > 3$: thus we see that the problem is impossible unless B's mass is at least three times as great as that of A.

(3) A number of balls, of the same given elasticity, A_1, A_2, A_3,.... are placed in a straight line: A_1 is projected with a given velocity so as to impinge on A_2; A_2 then impinges on A_3, and so on; to find the relation between the masses of the balls, in order that each of them may be at rest after impinging on the next; and to find the velocity of the n^{th} ball after its collision with the $(n-1)^{th}$.

Let u_1, u_2, u_3,... denote the velocities with which A_1, A_2, A_3,... impinge respectively on A_2, A_3, A_4,...., and suppose the masses of the balls to be represented by A_1, A_2, A_3...

Then, since the momentum of A_{n-1} before striking A_n must be the same as that of A_n after the collision,

$$A_n u_n = A_{n-1} u_{n-1} \text{ ....} (1).$$

Also, by the law of elasticity,

$$u_n = e u_{n-1} \text{ ....} (2).$$

From (2) we see that u_1, u_2, u_3,... form a geometrical progression of which e is the common ratio, u_n being therefore equal to $e^{n-1}u_1$.

Again, from (1) and (2), we see that

$$A_n = \frac{1}{e} A_{n-1}:$$

hence the masses of the balls are in geometrical progression, the common ratio being $\frac{1}{e}$.

(4) A, B, C, are three perfectly elastic balls at rest in the same straight line, B being between A and C: B is made to impinge directly upon A and, rebounding, strikes C: to shew that, if A and C, after having been struck by B, each move with the same velocity,

$$2m_2 = m_1 - m_3,$$

m_1, m_2, m_3, being the masses of A, B, C, respectively.

Let V be the initial velocity of B, and u_1, u_2, those of A, B, respectively, after their collision: see fig. (123).

Again, let u'_2, u_1, be the respective velocities of B, C, after their collision: see fig. (124).

Then, for the collision between A, B, the algebraical sum of the momenta being the same before and after collision,

$$m_1u_1 - m_2u_2 = m_2 V,$$

and, the relative velocity not being altered by the collision,

$$V = u_1 + u_2:$$

whence $$(m_1 - m_2) u_1 = 2m_2u_2 \quad\ldots\ldots\ldots\ldots\ldots\quad (1).$$

Again, for the collision between B, C,

$$m_2u_2 = m_2u'_2 + m_3u_1,$$

and $$u_2 = u_1 - u'_2:$$

whence $$2m_2u_2 = (m_2 + m_3) u_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (2).$$

From the equations (1) and (2), we see that

$$m_1 - m_2 = m_2 + m_3, \quad 2m_2 = m_1 - m_3.$$

(5) Two equal and imperfectly elastic spheres, A and B, collide directly; having given the original velocity of A, find that of B, in order that, after the blow, B may be at rest.

If u be the given velocity of A, B must meet it with a velocity equal to

$$\frac{1+e}{1-e} u.$$

(6) An inelastic sphere impinges directly upon another of twice its mass, which is at rest: prove that it loses two-thirds of its velocity by the collision.

(7) Two balls are moving in the same direction with the velocities 5, 7: after direct collision, their respective velocities are 6, 5: determine their mutual elasticity.

The required elasticity is $\frac{1}{2}$.

(8) A ball A impinges directly on a ball B, which is at rest, the balls being perfectly elastic: shew that, whatever be the masses, it is impossible for B to move after collision with twice A's original velocity.

(9) Two balls are moving in the same straight line, one of them only being acted on by a force; if the force be constant and tend towards the other ball, shew that the times which elapse between consecutive collisions decrease in geometrical progression.

(10) Three perfectly elastic balls, the masses of which are A, B, C, respectively, are placed with their centres in a straight line: A is made to impinge directly on B, and B consequently on C; compare the velocity thus communicated to C with the velocity which would have been communicated if the ball B had not been interposed.

The required ratio is equal to

$$\frac{2B(A+C)}{(A+B)(B+C)}.$$

(11) Equal beads, which are inelastic, are arranged at equal distances along a smooth horizontal wire: shew that, if one of them be projected with any velocity towards the next bead, the velocities over the different intervals are in harmonic progression, and that the time, before any proposed bead is put in motion, is to the time, in which it describes the next interval, as the initial distance of the bead, from the original position of the projected bead, is to double each interval.

(12) If in a set of five balls, of elasticity $\frac{1}{2}$, arranged in a straight line, which form a geometrical progression with a common ratio 2, the first impinge directly upon the second, the second upon the third, and so on; compare the velocity of the last ball with the original velocity of the first.

The required ratio is $\frac{1}{16}$.

(13) An elastic ball A, moving on a smooth horizontal plane, impinges directly on a ball B, of the same radius, at rest: shew that, if B afterwards impinges perpendicularly on a smooth wall, the original distance of which from the nearest point of B is given, the time, which elapses between the first and second collision of the balls, will be independent of their radius.

(14) A ball, of mass m_1, impinges directly upon a ball of mass m_2, which is at rest; if the vis viva before collision be λ times the vis viva after collision, find their common elasticity.

If e denote the elasticity,

$$e^2 = \frac{(1-\lambda)\,m_1 + m_2}{\lambda m_2}.$$

SECT. 2. *Oblique Collision.*

(1) A ball of mass m strikes another ball of mass nm, which is at rest, the inclination of the direction of the motion of the impinging ball to the line joining the centres of the balls, at the time of collision, being 30°: to find n in order that the impinging ball may, after collision, go off perpendicularly to the direction of its original motion.

Let V, V', be the velocities of the former ball before and after collision, v the velocity of the latter ball after collision.

Then, the component of the velocity of the impinging ball, at right angles to the line joining the centres, being the same before and after collision, we have

$$V \sin 30^\circ = V' \sin 60^\circ,$$

or

$$V = V' \sqrt{3} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, the algebraical sum of the momenta, resolved along the line joining the centres, being the same before and after collision,

$$mV \cos 30^\circ = mn \,.\, v - mV' \sin 30^\circ,$$

or

$$V\sqrt{3} = 2nv - V' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, the relative velocity of the two balls after collision being equal to e times that before collision,

$$eV\cos 30^\circ = v + V'\sin 30^\circ,$$

or
$$eV\sqrt{3} = 2v + V' \text{ .... (3)}.$$

From (1), (2), (3), it follows that

$$4 = n(3e - 1),$$

or
$$n = \frac{4}{3e - 1}.$$

This result shews that the problem is impossible if e be less than $\frac{1}{3}$. If $e = \frac{1}{3}$, $n = \infty$, and the motion of the impinging ball will be the same as if it had struck against an immoveable plane instead of a ball.

(2) Two equal billiard balls A and B, of given elasticity, are lying in contact on a table: in what direction must A be struck, so as, after colliding with B, to go off in a given direction?

Let e be the elasticity of the balls. Let V be the velocity which the blow upon A would have communicated to it had it been isolated, θ the inclination of V to the line joining the centres of the balls. Let u, v, be the components, after collision, of the velocity of A along and perpendicular to the line of the centres of the balls: u' the velocity of B after collision.

Then, by the formulæ of direct collision, we have, for the motion parallel to the line of centres,

$$V\cos\theta = u + u' \text{ .... (1)},$$

and
$$eV\cos\theta + u = u' \text{ .... (2)}.$$

From these two equations,

$$2u = (1 - e)V\cos\theta.$$

Also, since the velocity of A, at right angles to the line of centres, is the same before and after collision,

$$v = V\sin\theta.$$

Hence
$$\frac{v}{u} = \frac{2}{1 - e}\tan\theta.$$

But, if α denote the angle between the direction of A's motion and the line of centres,

$$\frac{v}{u} = \tan \alpha :$$

hence $$\tan \theta = \tfrac{1}{2}(1 - e) \tan \alpha.$$

COR. Let R denote the mutual action of the balls, during collision, estimated by the momentum added to B or subtracted from A: then

$$R = mu',$$

and therefore, by (1) and (2),

$$R = \tfrac{1}{2}(1 + e) V \cos \theta :$$

but, if P denote the blow on A, estimated by the momentum which it would have communicated to it if isolated, and m the mass of each ball,

$$P = mV :$$

hence $$R = \tfrac{1}{2}(1 + e) \cdot \frac{P \cos \theta}{m}.$$

(3) Two spheres, of masses m and m', moving, with their centres in one plane, with given equal velocities, come into collision, the directions of their motions before collision being inclined to each other at an angle α: to find the magnitude of the velocity of their centre of gravity after collision.

Let V denote the velocity of each sphere before collision, and θ, θ', the inclinations of the directions of their motions to the common tangent plane at their point of contact.

The algebraical sum of the momenta after collision is equal, in any assigned direction, to the algebraical sum before collision. Hence the sum of the momenta after collision, parallel to the tangent plane, is equal to

$$V(m \cos \theta + m' \cos \theta'),$$

and the algebraical sum, after collision, at right angles to the tangent plane, is equal to

$$V(m \sin \theta - m' \sin \theta').$$

Hence the components of the velocity of the centre of gravity after collision, in these two directions, are equal to

$$\frac{V}{m+m'} \cdot (m \cos\theta + m' \cos\theta') \text{ and } \frac{V}{m+m'} \cdot (m \sin\theta - m' \sin\theta');$$

and therefore its resultant velocity is equal to

$$\frac{V}{m+m'} \cdot \{m^2 + 2mm' \cos(\theta+\theta') + m'^2\}^{\frac{1}{2}}$$

$$= \frac{V}{m+m'} \cdot (m^2 + 2mm' \cos\alpha + m'^2)^{\frac{1}{2}}.$$

(4) Three equal smooth balls rest on a horizontal table, each being in contact with the other two; if one of them receive a blow in a given direction, at a given point in the plane passing through the centres of the balls, to determine the direction of its motion after colliding with the other two.

Let A, fig. (125), be the centre of the ball which receives the blow, and B, B', the centres of the other two, at the instant of impact. Let CAC' be the line passing through A and the point of contact of the balls B, B'. Let X, Y, be the components of the blow on A, resolved along and perpendicularly to CC'. We will first of all suppose the balls to be perfectly inelastic. Let R be the impulsive reaction between A and B, and R' between A and B'; let U, V, be the components of the velocity of A, after collision, along the line AC' and at right angles to it; and let u, u', be the velocities of B, B', respectively, after collision.

Then, for the motions of the balls B, B', we have

$$mu = R \text{ ....(1)},$$

$$mu' = R' \text{ ....(2)};$$

and, for the motion of A, resolving along and at right angles to AC',

$$mU = X - \frac{\sqrt{3}}{2} \cdot (R + R') \text{ ....(3)},$$

and $$mV = Y + \tfrac{1}{2}(R' - R) \text{ ....(4)}.$$

But, the balls being inelastic, the component of A's velocity along AB must be equal to the velocity of B, and the component of A's velocity along AB' must be equal to the velocity of B': hence

$$u = \tfrac{1}{2} U\sqrt{3} + \tfrac{1}{2} V \quad\text{......(5)},$$

and

$$u' = \tfrac{1}{2} U\sqrt{3} - \tfrac{1}{2} V \quad\text{......(6)}.$$

From (1), (3), (4), (5), we have

$$R = \frac{\sqrt{3}}{2}\{X - \frac{\sqrt{3}}{2}(R + R')\} + \tfrac{1}{2}\{Y + \tfrac{1}{2}(R' - R)\},$$

whence

$$4R + R' = X\sqrt{3} + Y \quad\text{......(7)}.$$

Similarly, from (2), (3), (4), (6), we should get

$$4R' + R = X\sqrt{3} - Y \quad\text{......(8)}.$$

From (7) and (8) we have

$$R + R' = \frac{2}{5} X\sqrt{3} \quad\text{......(9)},$$

and

$$R' - R = -\frac{2}{3} Y \quad\text{......(10)}.$$

If balls be elastic and e denote their elasticity, we must augment the values of R and R', as determined by (9) and (10), in the ratio of $1 : 1 + e$; and the equations (3) and (4) will then give the component velocities of A after impact.

Accordingly $$mU = X - \frac{\sqrt{3}}{2}(1 + e) \cdot \frac{2}{5} X\sqrt{3} = \frac{1}{5}(2 - 3e) X,$$

and $$mV = Y - \frac{1}{2}(1 + e)\frac{2}{3} Y \qquad = \frac{1}{3}(2 - e) Y.$$

Hence, if α be the inclination of the blow received by A, and θ the inclination of A's velocity after collision, to the line CC',

$$\tan\theta = \frac{5}{3} \cdot \frac{2 - e}{2 - 3e} \cdot \tan\alpha.$$

(5) A perfectly elastic ball A of any mass, lying on a smooth table, in contact with a perfectly hard vertical plane, is struck obliquely by a ball B of the same mass, which is moving on the table in a direction perpendicular to the vertical plane, α being the angle which the line of impact makes with the line of B's motion: to determine the direction of A's motion after collision, e being the mutual elasticity of A and B.

Let $m =$ the mass of each ball, $V =$ the impinging velocity of B. We will first suppose the balls to be perfectly inelastic. Let R, fig. (126), be the mutual impulse of the two balls, S the impulsive reaction of the vertical plane against the ball A. Let u be the velocity of A, after collision, at right angles to AB, and v the common velocity of A and B in the direction BA.

Then, for the motion of B, parallel to BA,

$$mv = mV\cos\alpha - R \quad\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

for the motion of A, parallel to BA,

$$mv = R - S\cos\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and for the motion of A, at right angles to BA,

$$mu = S\sin\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Since A, being inelastic, will not rebound from the vertical plane, we have

$$u\sin\alpha = v\cos\alpha, \quad\text{or}\quad u = v\cot\alpha \quad\ldots\ldots\ldots\ldots (4).$$

From (3) and (4),

$$mv = S\frac{\sin^2\alpha}{\cos\alpha},$$

and therefore, by (1) and (2),

$$S\frac{\sin^2\alpha}{\cos\alpha} = mV\cos\alpha - R,$$

$$S\frac{\sin^2\alpha}{\cos\alpha} = R - S\cos\alpha;$$

adding together these two equations, we have

$$2S\frac{\sin^2\alpha}{\cos\alpha} = mV\cos\alpha - S\cos\alpha,$$

and therefore $$S=\frac{mV\cos^2\alpha}{1+\sin^2\alpha};$$

hence also

$$R=S\left(\cos\alpha+\frac{\sin^2\alpha}{\cos\alpha}\right)=\frac{mV\cos\alpha}{1+\sin^2\alpha}.$$

Next, suppose the mutual elasticity of A and B to be e, that between A and the vertical plane being unity. Let u', v', be the components of A's velocity, after collision, at right angles and parallel to BA. Then, R and S being now replaced by $(1+e)R$ and $2S$, we have, for A's motion,

$$mv'=(1+e)\cdot\frac{mV\cos\alpha}{1+\sin^2\alpha}-2\frac{mV\cos^2\alpha}{1+\sin^2\alpha}\cos\alpha,$$

or $$v'=\frac{V\cos\alpha}{1+\sin^2\alpha}(1+e-2\cos^2\alpha),$$

and $$mu'=2S\sin\alpha=\frac{2mV\sin\alpha\cos^2\alpha}{1+\sin^2\alpha},$$

or $$u'=\frac{2V\sin\alpha\cos^2\alpha}{1+\sin^2\alpha}.$$

Hence, $$\frac{v'}{u'}=\frac{1+e-2\cos^2\alpha}{2\sin\alpha\cos\alpha}.$$

Let $\theta=$ the inclination of A's motion, after collision, to the normal to the vertical plane: then

$$\tan\theta=\frac{u'\cos\alpha+v'\sin\alpha}{u'\sin\alpha-v'\cos\alpha}$$

$$=\frac{2\sin\alpha\cos^2\alpha+\sin\alpha(1+e-2\cos^2\alpha)}{2\sin^2\alpha\cos\alpha-\cos\alpha(1+e-2\cos^2\alpha)}$$

$$=\frac{(1+e)\sin\alpha}{(1-e)\cos\alpha},$$

whence $$\theta=\tan^{-1}\left(\frac{1+e}{1-e}\tan\alpha\right).$$

(6) One perfectly elastic ball impinges upon another, which is at rest: if the original direction of the striking ball is inclined at an angle of 45^0 to the line joining their centres, find the angle between the directions of its motion before and after collision.

If m be the mass of the impinging ball and m' of the other ball, the required angle is equal to

$$\tan^{-1}\left(\frac{m'}{m}\right).$$

(7) Two equal balls, the common elasticity of which is e, start at the same instant with equal velocities from the opposite angles of a square along two contiguous sides and collide; determine the angle between their directions of motion after collision.

The required angle is equal to

$$\tfrac{1}{2}\tan^{-1}\left(\frac{2e}{1-e^2}\right).$$

(8) A and B are two equal and perfectly elastic spheres; A, moving with a given velocity V, impinges on B, which is at rest, the direction of A's motion, before collision, making an angle of 60^0 with the straight line which joins their centres at the instant of collision: determine the directions and velocities of A and B after collision.

A moves, after collision, with a velocity $\frac{1}{2}V\sqrt{3}$, at right angles to the initial position of the line of centres, and B moves along this line with a velocity $\frac{1}{2}V$.

(9) The centres of two balls A and B, the common elasticity of which is $\frac{1}{2}$, are moving, in one plane, with equal velocities, in directions which make angles of 60^0 with the line through their centres on the same side of the line, at the instant of collision: compare their masses, when A's motion after collision is perpendicular to the line through their centres, and find the distance between the balls two seconds after collision.

The mass of A must be twice that of B, and, if V be the velocity of each ball before collision, the distance between the balls, two seconds after collision, will be equal to V.

(10) A ball A impinges on an equal ball B, which is at rest: supposing the velocities of the two balls after collision to be equal, determine the change of direction in the motion of A.

If e denote the elasticity of the balls, the angle between the directions of A's motion before and after collision is equal to $\tan^{-1}(e^{\frac{1}{2}})$.

(11) A ball A impinges obliquely on another ball B, which is at rest, and, after collision, the directions of motion of A and B make equal angles with A's previous motion: find these angles.

If m, m', be the masses of A, B, respectively, e their mutual elasticity, and θ either of the required angles,

$$\tan^2\theta = \frac{(1+2e)\,m' - m}{m + m'}.$$

(12) An imperfectly elastic ball lies on a billiard-table: determine the direction in which an equal ball must strike it, in order that they may impinge upon a side of the table at equal given angles.

If e be the elasticity of the balls, α the magnitude of each of the given angles, fig. (127), and ϕ the inclination of the direction of the motion of the impinging ball to the line of centres of the balls at the instant of collision,

$$\tan\phi = \frac{1}{2}(1-e)\tan 2\alpha.$$

(13) A ball, moving on a smooth horizontal table, impinges on two others in all respects like it, which are lying in contact, at the same moment: determine the mutual elasticity of the balls, if the impinging one be brought to rest.

The elasticity is equal to $\frac{2}{3}$.

(14) Two balls, of equal volumes and masses, moving with equal velocities, in directions passing through the centre of a third ball, which is at rest, impinge upon it, and upon one another simultaneously: compare the mass of either of the impinging balls with that of the third ball in order that, after collision, the

directions of motion of the two impinging balls may be perpendicular to that of the third, the coefficient of elasticity being $\frac{1}{2}$.

The mass of the third ball must be four times as great as that of either of the impinging balls.

(15) Three equal and perfectly elastic balls A, B, C, move with equal velocities towards the same point, in directions equally inclined to each other; suppose, first, that they impinge upon each other at the same instant; secondly, that B and C impinge on each other, and immediately afterwards simultaneously on A; and, thirdly, that B and C impinge simultaneously on A just before touching each other; and let V_1, V_2, V_3, be the velocities of A after impact on these suppositions respectively: prove that

$$V_2 = \frac{1}{5} V_1, \text{ and that } V_3 = \frac{7}{5} V_1.$$

CHAPTER VI.

IMPACT OF FALLING BODIES.

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(1) FROM what height must a weight of 12 pounds fall in order that it may impinge upon the ground with the same effect as a weight of 25 pounds falling 6 feet?

The required height is $26\frac{1}{24}$ feet.

(2) Two inelastic bodies are dropped from P, Q, on to a smooth inclined line XY, and reach the lowest point of the line with the same velocity: prove that PQ is perpendicular to XY.

(3) Two perfectly elastic balls are dropped from two points not in the same vertical line, and strike against a perfectly elastic horizontal plane: shew that their centre of gravity will never reascend to its original height, unless the initial heights of the balls be in the ratio of two square whole numbers.

(4) Two equal perfectly elastic balls are dropped at the same instant from altitudes

$$h = \frac{1}{2}g, \quad h' = \frac{9}{2}g,$$

above a horizontal table: prove that, at the end of $6n \pm 1$ seconds, n being any positive integer, the velocity of the centre of gravity suddenly changes from g to 0, or from 0 to g.

If $h' = \frac{(3600)^2}{2p^2} \cdot g$, p being any positive integer, prime to 2, 3, and 5, prove that it will be exactly two hours before the centre of gravity attains its original altitude.

CHAPTER VII.

IMPACT OF PROJECTILES.

SECT. 1. *Impact in the plane of motion.*

(1) THERE are two parallel walls, the distance between which is equal to their height: from the top of one of them a perfectly elastic ball is thrown horizontally, so as to fall at the foot of the same wall, after rebounding from the other: to determine the position of the focus of the first path.

Let h be the height of each wall, u the velocity of projection, and t the time of flight between the instant of projection and the return of the ball to the foot of the first wall.

Then, the vertical descent not being affected by the impact at the second wall,

$$h = \frac{1}{2} g t^2 \qquad\qquad (1).$$

Again, the horizontal velocity being the same in magnitude throughout the motion, and h being the distance between the walls,

$$h = \frac{1}{2} u t \qquad\qquad (2).$$

From (1) and (2) we see that

$$u^2 = 2gh \qquad\qquad (3).$$

The equation (3) shews that the height of the directrix of the first path above the point of projection is equal to h: hence, the point of projection being the vertex of the parabola, the focus must be at the foot of the first wall.

(2) A heavy particle, of elasticity e, is projected, at an angle of inclination β to a given inclined plane of altitude h, from the foot of the plane, so that after one rebound it just reaches the top of the plane, and describes part of the descending branch of the parabola after passing the top: to determine the latus rectum of the parabola.

Let V be the velocity of projection, and t, t', the times of the first and second bounds.

Then, since, by the nature of the problem, the velocity of the particle along the plane is zero at the top of the plane, we have

$$V\cos\beta = g\sin\alpha\,(t+t') \text{ .... (1).}$$

Again, since $V\sin\beta$, $eV\sin\beta$, are the velocities, normal to the plane, at the commencement of the first and second bound, respectively, we have also

$$t = \frac{2V\sin\beta}{g\cos\alpha},\quad t' = \frac{2eV\sin\beta}{g\cos\alpha},$$

and therefore, by (1),

$$\cot\beta = 2\,(1+e)\tan\alpha \text{ .... (2).}$$

Again, when the particle leaves the plane, its horizontal velocity is equal to

$$eV\sin\beta\,.\,\sin\alpha:$$

hence the latus rectum is equal to

$$\frac{2e^2V^2\sin^2\beta\sin^2\alpha}{g}:$$

but, since the particle only just reaches the top of the plane,

$$V^2 = 2gh:$$

hence the latus rectum is equal to

$$\begin{aligned} &4e^2h\sin^2\beta\sin^2\alpha \\ &= 4e^2h\sin^2\beta\,.\,\frac{\tan^2\alpha}{1+\tan^2\alpha} \\ &= \frac{4e^2h\cos^2\beta}{4(1+e)^2+\cot^2\beta},\ \text{ by (2).} \end{aligned}$$

(3) A parabola is placed with its axis vertical and vertex downwards, and a perfectly elastic ball, dropped vertically, strikes the parabola with the velocity acquired in falling freely from rest through a distance equal to one-fourth of the latus rectum: to find where it must strike the parabola, that after reflection it may pass through the vertex.

Let V be the velocity of impact, A the vertex, S the focus, and P the required point. Let t be the time between leaving P and arriving at A.

Then, by the hypothesis,

$$V^2 = 2g \,.\, AS \text{..........................} (1).$$

Since the direction of incidence and the line PS make equal angles with the tangent at P, the ball, after striking P, will be reflected in the direction PS with the velocity V.

Now PS is the space due to the velocity of reflection, and SA that due to the action of gravity, in the time t: hence

$$PS = Vt, \quad AS = \frac{1}{2} gt^2,$$

and therefore $$\frac{PS^2}{AS} = \frac{2V^2}{g} = 4AS, \text{ by (1)}:$$

hence $$PS = 2AS:$$

this result shews that P is one end of the latus rectum.

(4) A ball, of elasticity e, is projected from a point in an inclined plane, and, after n rebounds, returns to its point of projection: to prove that, α being the inclination of the plane, and β the angle between the direction of projection and the plane,

$$\cot\alpha \,.\, \cot\beta = \frac{1 - e^{n+1}}{1 - e}.$$

Considering only motion normal to the plane, we see that, V being the velocity of projection, the times of the several paths are

$$\frac{2V\sin\beta}{g\cos\alpha}, \quad \frac{2eV\sin\beta}{g\cos\alpha}, \quad \frac{2e^2V\sin\beta}{g\cos\alpha}, \ldots \frac{2e^nV\sin\beta}{g\cos\alpha},$$

and therefore that the whole time between leaving and returning to the point of projection is equal to

$$\frac{2V\sin\beta}{g\cos\alpha} \cdot \frac{1 - e^{n+1}}{1 - e} \text{..........................} (1).$$

Again, considering motion only parallel to the plane, the same interval is equal to

$$\frac{2V\cos\beta}{g\sin\alpha} \text{................................} (2).$$

Hence, equating the expressions (1) and (2), we have

$$\cot\alpha \,.\, \cot\beta = \frac{1 - e^{n+1}}{1 - e}.$$

(5) A perfectly elastic ball is projected from a point in a

vertical wall so as to strike against another vertical wall, parallel to the former: to determine the angle of projection in order that, if the velocity of projection be that due to n times the distance between the walls, it may return to the point of projection or the point opposite to it in the other wall, after n bounds.

Let V denote the velocity and α the angle of projection; c the distance between the walls. Then, $V\cos\alpha$ being always the horizontal velocity of the ball, the time of n bounds must be equal to

$$\frac{nc}{V\cos\alpha}.$$

Again, $V\sin\alpha$ being the vertical component of the velocity of projection, the time which elapses before the ball returns to the level of the point of projection will be equal to

$$\frac{2V\sin\alpha}{g}.$$

Since these two times are, by the hypothesis, equal, we have

$$ncg = V^2 \sin 2\alpha:$$

but, by the supposition, $V^2 = 2ncg$: hence

$$\sin 2\alpha = \frac{1}{2}, \quad \alpha = \frac{\pi}{12}.$$

(6) A ball, of elasticity e, is projected upwards from a point in a plane, which is inclined to the horizon at an angle α, the velocity of the ball's projection making an angle β with the plane, and the plane of projection being perpendicular to the intersection of the inclined plane with the horizon: to find the condition that the ball may cease to hop at the highest point of the plane which it reaches.

If V be the velocity of projection, the times of describing the infinite series of successive bounds are

$$\frac{2V\sin\beta}{g\cos\alpha}, \quad \frac{2eV\sin\beta}{g\cos\alpha}, \quad \frac{2e^2V\sin\beta}{g\cos\alpha}, \ldots\ldots,$$

and the sum of these times is equal to

$$\frac{2V\sin\beta}{g\cos\alpha} \cdot \frac{1}{1-e}.$$

But the velocity along the plane must be zero at the end of these times, since the ball must cease to ascend the plane when the normal velocity of rebound becomes zero: hence

$$0 = V\cos\beta - g\sin\alpha \, . \, \frac{2V\sin\beta}{g(1-e)\cos\alpha},$$

and therefore we have, for the required condition,

$$\tan\alpha \, . \, \tan\beta = \frac{1}{2}(1-e).$$

(7) A ball is projected from a point in an inclined plane, and, after impinging once upon the plane, rises vertically; and finally ceases to hop at the point of projection: to determine the elasticity of the ball.

Let e be the elasticity of the ball, and v, f, the components of the velocity of projection and of gravity, respectively, at right angles to the plane.

Let t be the time between first and second impact, t' that between second impact and the return to the point of projection, and T the time between first impact and the return to the point of projection.

Then $$T = t + t' \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now $$t = \frac{2ev}{f} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also, since the ball rises vertically after the first impact, the point of second impact must coincide with that of first impact, and therefore t' must be equal to the time between projection and first impact: hence

$$t' = \frac{2v}{f} \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again, since the ball ceases to hop at the end of the time T after first impact,

$$T = \frac{2ev}{f}(1 + e + e^2 + \ldots) = \frac{2ev}{(1-e)f} \quad\ldots\ldots\ldots(4).$$

From (1), (2), (3), (4), there is

$$\frac{e}{1-e} = e + 1, \quad 1 - e^2 = e,$$

and therefore $$e = \frac{1}{2}(\sqrt{5} - 1).$$

(8) A ball, of given elasticity, drops from a given height upon a given inclined plane: to find the range on the plane at the first and second rebound.

Let α denote the inclination of the plane to the horizon and V the velocity of the ball the instant before impact.

Let A, B, C, be the three successive points of impact of the ball on the plane. Let e be the ball's elasticity.

The components of the velocity, just after impact at A, will be $V \sin\alpha$, along the plane, and $eV\cos\alpha$, perpendicularly to it. Hence the time of the first range will be equal to

$$\frac{2eV\cos\alpha}{g\cos\alpha} = \frac{2eV}{g},$$

and therefore

$$AB = \frac{2eV}{g} \cdot \left(V\sin\alpha + \tfrac{1}{2}g\sin\alpha \cdot \frac{2eV}{g}\right)$$

$$= \frac{2e}{g}(1+e)\,V^2\sin\alpha.$$

Again, the component of the ball's velocity, perpendicularly to the plane, the instant after impact at B, is equal to $e^2V\cos\alpha$, and therefore the time of the second range is equal to

$$\frac{2e^2V\cos\alpha}{g\cos\alpha} = \frac{2e^2V}{g}.$$

Hence the time of moving from A to C is equal to

$$2e(1+e) \cdot \frac{V}{g},$$

and therefore

$$AC = 2e(1+e) \cdot \frac{V}{g} \cdot \left\{V\sin\alpha + \tfrac{1}{2}g\sin\alpha \cdot \frac{2e(1+e)V}{g}\right\}$$

$$= \frac{2e(1+e)V^2}{g} \cdot \sin\alpha \cdot (1+e+e^2),$$

and consequently

$$BC = \frac{2e^2}{g}(1+e)^2V^2\sin\alpha.$$

If h be the height through which the ball drops before striking the plane,

$$V^2 = 2gh:$$

hence $$AB = 4e\,(1+e)\,h \sin\alpha,$$

and $$BC = 4e^2(1+e)^2 h \sin\alpha.$$

(9) An imperfectly elastic ball is projected, at an elevation of 45°, against a smooth vertical wall, the motion taking place in a vertical plane perpendicular to the wall: after impact, the ball strikes the ground between the wall and the point of projection, and, rebounding once, reaches the point of projection: to prove that, if a be the distance of the point of projection from the wall, and b be the height, above the ground, of the point where the ball strikes the wall, then

$$b = a \,.\, \frac{1-ee'}{1+e};$$

where e is the relative elasticity of the ball and the wall, and e' that of the ball and the ground, which is supposed perfectly smooth.

Let A, fig. (128), be the point of projection, P the point where the ball strikes the wall CD, B the point where it first strikes the ground AC.

Let $V =$ the velocity of projection. Then, the horizontal component of V being $\dfrac{V}{\sqrt{2}}$,

$$\text{the time through } AP = \frac{a\sqrt{2}}{V};$$

and, the vertical component of V being $\dfrac{V}{\sqrt{2}}$,

$$\text{the time through } APB = \frac{V\sqrt{2}}{g};$$

and therefore

$$\text{the time through } PB = \frac{V\sqrt{2}}{g} - \frac{a\sqrt{2}}{V} \,\ldots\ldots\ldots\, (1).$$

Again, the horizontal component of the ball's velocity, just after striking P, being $\dfrac{eV}{\sqrt{2}}$,

$$\text{the time through } PBA = \frac{a\sqrt{2}}{eV};$$

and, the vertical component of the ball's velocity, just after striking B, being $\frac{e'V}{\sqrt{2}}$,

$$\text{the time through } BA = \frac{e'V\sqrt{2}}{g};$$

and therefore

$$\text{the time through } PB = \frac{a\sqrt{2}}{eV} - \frac{e'V\sqrt{2}}{g} \text{ .... (2).}$$

From (1) and (2) it follows that

$$\frac{V\sqrt{2}}{g} - \frac{a\sqrt{2}}{V} = \frac{a\sqrt{2}}{eV} - \frac{e'V\sqrt{2}}{g},$$

and therefore $$V^2 = \frac{ag(1+e)}{e(1+e')} \text{ .... (3).}$$

Finally, t being the time through AP,

$$\begin{aligned} b &= \frac{1}{\sqrt{2}} Vt - \tfrac{1}{2}gt^2 \\ &= \frac{1}{\sqrt{2}} V . \frac{a\sqrt{2}}{V} - \tfrac{1}{2}g . \left(\frac{a\sqrt{2}}{V}\right)^2 \\ &= a - \frac{a^2 g}{V^2} \\ &= a\left\{1 - \frac{e(1+e')}{1+e}\right\}, \quad \text{by (3),} \\ &= a . \frac{1-ee'}{1+e}. \end{aligned}$$

(10) From a point P on the ground, equidistant between two vertical planes A and B, an imperfectly elastic ball is projected, with a velocity $(2gh)^{\frac{1}{2}}$, towards A and reflected by it to B; find c, the altitude of the highest point of B the ball can reach: and shew, first, that, if α be the elevation of the direction of projection which enables the ball to attain that altitude,

$$\tan^2 \alpha = \frac{h}{h-c};$$

secondly, that, if α', α'', be two elevations such that

$$\tan \alpha' + \tan \alpha'' = 2 \tan \alpha,$$

two balls, projected in those directions towards A, will hit the

same point of B; and, thirdly, that, if $\sin 2\alpha''' > \cot \alpha$, a ball, projected in a direction of which the elevation is α''', will hit some point of B, otherwise not.

Let θ be the angle of projection, t the time of flight from P to B; τ, τ', the times from P to A, and from A to B respectively; x the altitude of the point of impact of the ball on B.

Then, $2a$ being the distance between A and B,

$$x = (2gh)^{\frac{1}{2}} \sin\theta \,.\, t - \tfrac{1}{2}gt^2 \quad\text{.................}\quad (1),$$

$$a = (2gh)^{\frac{1}{2}} \cos\theta \,.\, \tau \quad\text{.........................}\quad (2),$$

$$2a = e\,(2gh)^{\frac{1}{2}} \cos\theta \,.\, \tau' \quad\text{......................}\quad (3),$$

and

$$t = \tau + \tau' \quad\text{....................................}\quad (4).$$

From (2), (3), (4), putting $\dfrac{2+e}{e} = \lambda$, we have

$$t = \frac{a}{(2gh)^{\frac{1}{2}} \cos\theta} \cdot \frac{2+e}{e} = \frac{\lambda a}{(2gh)^{\frac{1}{2}} \cos\theta},$$

and therefore, from (1),

$$x = \lambda a \tan\theta - \frac{\lambda^2 a^2}{4h\cos^2\theta} \quad\text{..................}\quad (5),$$

$$\tan^2\theta - \frac{4h}{\lambda a}\tan\theta + \frac{4hx}{\lambda^2 a^2} + 1 = 0 \quad\text{............}\quad (6),$$

$$\left(\tan\theta - \frac{2h}{\lambda a}\right)^2 + \frac{4hx}{\lambda^2 a^2} + 1 - \frac{4h^2}{\lambda^2 a^2} = 0.$$

From this equation we see that, c being the greatest value of x and α the corresponding value of θ,

$$\tan\alpha = \frac{2h}{\lambda a},$$

and

$$\frac{4hc}{\lambda^2 a^2} = \frac{4h^2}{\lambda^2 a^2} - 1, \quad c = h - \frac{\lambda^2 a^2}{4h},$$

and therefore

$$\tan^2\alpha = \frac{h}{h-c}.$$

Again, the equation (6) shews that, for a single value of x, there are two values of θ: let α', α'', be these two values: then, by a property of quadratic equations,

$$\tan\alpha' + \tan\alpha'' = \frac{4h}{\lambda a} = 2\tan\alpha.$$

Also, unless x, as determined by (5), be positive, the ball will strike the ground before reaching B: hence, α''' being a proper value of θ, to insure impact on B, we must have

$$\lambda a \tan \alpha''' > \frac{\lambda^2 a^2}{4h \cos^2 \alpha'''},$$

$$\sin 2\alpha''' > \frac{\lambda a}{2h}, \quad \text{or } \sin 2\alpha''' > \cot \alpha.$$

(11) A ball, of elasticity $\frac{1}{2}$, is thrown against a smooth vertical wall: find the direction in which it may be projected with the least velocity, so that it shall return to the point of projection.

The angle of projection must be $\frac{\pi}{4}$.

(12) Determine the velocity with which a ball of given elasticity must be projected from a given point, in a given direction, towards a vertical wall, in order that, after striking the wall, it may return to the point of projection.

If e denote the elasticity, c the distance of the point of projection from the wall, and α the angle of projection, the required velocity is equal to

$$\left\{\frac{gc\,(1+e)}{e \sin 2\alpha}\right\}^{\frac{1}{2}}.$$

(13) From the highest point A of a vertical circle, a chord AP is drawn, making an angle θ with the vertical diameter AB: a perfectly elastic ball impinges on the concave circumference at P in the direction AP, with the velocity due to sliding down AP: prove that if, after rebounding from the curve, it passes through B,

$$\sin \theta + 2 \sin 5\theta = 0.$$

(14) A perfectly elastic particle, after falling through a height h, strikes the arc of a parabola, the latus rectum of which is l, at a point the distance of which from the directrix is a, the axis of the parabola being vertical and its vertex down-

wards: prove that, if $a^2 = lh$, the particle will return to the point of projection by retracing its whole course.

(15) An imperfectly elastic ball is projected from a point between two vertical planes, the plane of motion being perpendicular to both; shew that the arcs described between the rebounds are portions of parabolas the latera recta of which are in geometrical progression.

(16) A deep well has two sides parallel to one another: a ball of given elasticity is thrown so as to impinge normally on one of these sides and rebound from side to side: determine the depths of the points of successive impact.

Let $a =$ the distance between the two sides of the well, $e =$ the elasticity, $u =$ the velocity of incidence at the first impact. Then the depth of the point of n^{th} impact, below the point of first impact, is equal to

$$\frac{a^2 g}{2u^2} \cdot \left(\frac{e^{1-n} - 1}{1 - e}\right)^2.$$

(17) A ball is projected from a point in a horizontal plane, and makes one rebound: find the relation between the angle of projection and the elasticity, in order that the second range may be equal to the greatest height which the ball attains.

If α be the angle of projection and e the elasticity,

$$\tan \alpha = 4e.$$

(18) A perfectly elastic particle, dropped from a point P, impinges on an inclined plane at Q: if PN be perpendicular to the plane, prove that the range is equal to $8\,QN$, and that, if the lowest point of the range be given, the locus of P is a straight line, through this lowest point, inclined to the plane at an angle

$$\tan^{-1}\left(\frac{1}{9}\cot \alpha\right),$$

α being the inclination of the plane to the horizon.

(19) A perfectly elastic ball is dropped from a height h above an indefinite horizontal plane, and, after falling through a space x, is reflected at a small plane inclined to the horizon

at an angle α, which is not greater than $\frac{\pi}{6}$: determine the value of x in order that the horizontal distance described by the ball before striking the horizontal plane may be a maximum.

The required value of x is $\frac{1}{4}h \operatorname{cosec}^2 \alpha$.

(20) From what point in a given horizontal plane must a ball of given elasticity be let fall upon a given inclined plane, in order that it may strike a given point in that plane after one rebound?

Let α be the inclination of one plane to the other, a the distance of the given point and x of the required point from the intersection of the planes: then, e denoting the elasticity,

$$x = \frac{a \cos \alpha}{1 + 4e(1+e)\sin^2 \alpha}.$$

(21) From A, a point in a vertical circle, of which the centre is O, a perfectly elastic ball is dropped: it rebounds from B to C, and thence vertically to D, after which it returns by the same path, B, C, D, being points in the circle: determine the angle BOC.

The angle BOC is defined by the equation

$$\cos \angle BOC = \frac{\sqrt{2} - 1}{2}.$$

(22) A ball is projected from a point A in a horizontal plane AB, so as to strike a vertical wall BC perpendicularly at C: shew that, if the elasticity of the ball be equal to $\frac{1}{2}$, relatively both to the wall and to the plane, it will come to A after one rebound on AB.

(23) A heavy ball is thrown horizontally from a point A so as to hit a point B, after one rebound at a point C of a horizontal plane: supposing e to be the coefficient of elasticity, and the height of B from the plane to be e^2 times that of A, the height of A being such that a body would drop from it to the plane in $1''$, shew that the point where the ball must hit the ground divides the horizontal distance between A and B into two parts which are as 1 to e.

(24) An elastic ball is projected from a point in a plane, which is inclined to the horizon at an angle α, the direction of projection making an angle β with the upward direction of the inclined plane: at the end of one rebound it shoots through a horizontal bore at a point in the plane, just large enough to admit it: prove that

$$\cot\beta = 2(1+e)\tan\alpha + e\cot\alpha.$$

(25) OA, OB, are two planes inclined at angles α, β, to the horizon: a perfectly elastic ball is projected from A and strikes B, and continues to rebound along the same curvilinear path between A and B: having given the length of the chord AB, find its inclination to the horizon and the time of flight.

If c be the length of the chord AB, γ its inclination to the horizon, and t the time of flight, then

$$\tan\gamma = \tfrac{1}{2}(\cot\beta - \cot\alpha),$$

$$t^2 = \frac{c}{g} \cdot \frac{\sin(\alpha+\beta)}{\{4\sin^2\alpha\sin^2\beta + \sin^2(\alpha-\beta)\}^{\frac{1}{2}}}.$$

(26) From what height must a perfectly elastic ball be let fall into a fixed hemispherical bowl, in order that it may rebound horizontally at the first impact and strike the lowest point of the bowl at the second?

If r be the radius of the bowl, the required height is equal to $\dfrac{r}{4(2-\sqrt{2})}$.

(27) A perfectly elastic ball is projected with a given velocity from a point between two parallel walls, and returns to the point of projection, after being once reflected at each wall: prove that its angle of projection is either of two complementary angles.

(28) A ball, of elasticity e, is projected from a point in a horizontal plane: if the distance of the point of n^{th} impact be equal to four times the sum of the greatest altitudes of the ranges, determine the angle of projection.

The required angle is equal to

$$\tan^{-1}\left(\frac{1+e}{1+e^n}\right).$$

(29) A ball is projected from a point A, in a horizontal line AB, at an elevation of 45^0 against a wall BC, and in a vertical plane perpendicular to the wall: after impact at C, it strikes the ground between A and B, and arrives at A after n rebounds: find the ratio of BC to AB in terms of the elasticity of the ball, which is supposed to be the same in relation both to the wall and to the ground.

The required ratio is equal to

$$1-\frac{e-e^{n+2}}{1-e^2}.$$

(30) A perfectly elastic ball is projected at an inclination β to a plane inclined to the horizon at an angle α, so as to ascend it by bounds: find the condition that the ball may rise vertically at the n^{th} bound.

The required condition is expressed by the equation

$$\cot\alpha \,.\, \cot\beta = 2n+1.$$

(31) A perfectly elastic ball is projected in a direction perpendicular to an inclined plane: prove that the ranges of the successive hops on the plane are in arithmetical progression, and that one straight line will be a tangent to all the curves described.

(32) A ball, of elasticity e, is projected with a velocity V at an angle β to the upward direction of a plane inclined to the horizon at an angle α, and in a plane perpendicular to the intersection of the inclined plane with the horizon: prove that the ball will cease to hop at a distance a from the point of projection, provided that

$$a=\frac{V^2}{g}\,.\,\frac{\sin 2\beta \,.\, \sec\alpha}{1-e}\,.\left(1-\frac{\tan\alpha \,.\, \tan\beta}{1-e}\right).$$

(33) A ball, of elasticity e, is projected with a certain velocity at an angle β to the upward direction of a plane inclined to the

horizon at an angle α: find the condition that the ball may arrive at the point of projection and cease hopping simultaneously.

The required condition is expressed by the equation

$$\tan\alpha \,.\, \tan\beta = 1 - e.$$

(34) A ball, of elasticity e, is projected from the foot of a smooth plane, inclined at an angle α to the horizon, in a direction making an angle β with the upward direction of the plane: prove that it will in the n^{th} bound move up or down the plane accordingly as

$$\cot\alpha \cot\beta > \text{ or } < \frac{2 - e^{n-1} - e^n}{1 - e}.$$

SECT. 2. *Impact in any direction.*

(1) An imperfectly elastic sphere is projected with a velocity V and impinges against a given vertical plane, which makes an angle β with the plane of the sphere's motion, a being the distance of the point of projection from the given plane, and α the inclination of V's direction to the horizon: to find the velocity V' of the sphere just after impact; and to determine the values of α and V', when the elasticity is perfect, in order that V' may be a minimum.

If t be the time between the instants of projection and impact,

$$\frac{a}{\sin\beta} = V\cos\alpha \,.\, t \quad\text{.....................}\quad (1).$$

The normal velocity of impact is equal to $V\cos\alpha\sin\beta$, and therefore the normal velocity of resilition is equal to

$$eV\cos\alpha\sin\beta.$$

Again, the vertical velocity at the instant of impact is equal to

$$V\sin\alpha - gt = V\sin\alpha - \frac{ag}{V\sin\beta\cos\alpha}, \text{ by (1)}.$$

Also, the horizontal velocity, parallel to the given vertical plane, at the instant of impact, is equal to $V\cos\alpha\cos\beta$.

Hence

$$V'^2 = e^2V^2\cos^2\alpha\sin^2\beta + \left(V\sin\alpha - \frac{ag}{V\cos\alpha\sin\beta}\right)^2 + V^2\cos^2\alpha\cos^2\beta.$$

If $e=1$, we see that

$$V'^2 = V^2 + \frac{a^2g^2}{V^2\cos^2\alpha\sin^2\beta} - \frac{2ag\tan\alpha}{\sin\beta}$$

$$= V^2 + \frac{a^2g^2}{V^2\sin^2\beta} + \frac{a^2g^2}{V^2\sin^2\beta}\,.\tan^2\alpha - \frac{2ag\tan\alpha}{\sin\beta}$$

$$= \frac{a^2g^2}{V^2\sin^2\beta} + \left(\frac{ag\tan\alpha}{V\sin\beta} - V\right)^2.$$

From this result we see that V' is least when

$$\tan\alpha = \frac{V^2\sin\beta}{ag},$$

the corresponding value of V' being given by the equation

$$V' = \frac{ag}{V\sin\beta}.$$

(2) A perfectly elastic ball is projected with a given velocity from the middle point of one of the sides of a given equilateral three-cornered room: it strikes the two other sides and returns to the point of projection: to determine the direction in which it was projected.

Let ABC, fig. (129), be the floor of the room, PQR the projection of the path of the ball upon the floor, P being that of the point whence the ball started.

It is easily seen that the angles indicated in the figure by θ are equal, and that those indicated by ϕ are also equal.

Now $$PB\sin\frac{\pi}{3} = PR\,.\sin\theta,$$

$$PC\sin\frac{\pi}{3} = PQ\,.\sin\phi,$$

and therefore, since $PB = PC$,

$$PR\,.\sin\theta = PQ\,.\sin\phi:$$

but, from the triangle PQR,

$$PR : PQ :: \sin 2\phi : \sin 2\theta;$$

hence $$\sin 2\phi \sin\theta = \sin 2\theta \,.\, \sin\phi,$$

$$\cos\phi = \cos\theta,$$

and therefore $$\phi = \theta = \tfrac{1}{3}\pi.$$

Hence, as is easily seen, the triangle PQR is equilateral, each side being equal to half a side of ABC.

Let $V =$ the velocity of projection, $\alpha =$ the inclination of the direction of projection to the horizon, and $a =$ the length of a side of ABC.

Since the lines PQ, QR, RP, are all equal, the times through the arcs, of which they are the projections, must be equal: let t denote the time through each.

Then, $V\sin\alpha$ being the vertical component of the velocity of projection and t the whole time of flight,

$$3t = \frac{2V\sin\alpha}{g},$$

and, $\frac{1}{2}a$ being the length of each side of the triangle PQR,

$$\tfrac{1}{2}a = V\cos\alpha \,.\, t.$$

Hence $$\frac{3}{2}a = \frac{V^2 \sin 2\alpha}{g},$$

and therefore $$\alpha = \tfrac{1}{2}\sin^{-1}\left(\frac{3ag}{2V^2}\right).$$

(3) Three planes, two vertical and one horizontal, are mutually at right angles: a ball is projected from a given point in the horizontal plane, with a given velocity and in a given direction, and strikes successively each of the two vertical planes: to find the point at which it will again strike the horizontal plane, the coefficient of elasticity being supposed to be the same between the ball and each of the vertical planes.

Let OA, OB, OC, fig. (130) be the intersections of the three planes, OC being vertical, and OA, OB, horizontal: let E

be the point of projection, $EFGH$ the path of the ball, H being the point at which it again strikes the plane AOB.

Draw EN at right angles to OB, and HM to OA. Let $EN=a$, $ON=b$, $OM=x$, $HM=y$. Let u, v, w, be the components of the velocity of projection parallel to AO, BO, OC, respectively. Let t_1, t_2, t_3, be the times of flight through the portions EF, FG, GH, of the path, respectively. Let e represent the elasticity.

The velocity, parallel to AO, from E to F, is u; hence

$$a = ut_1 \qquad\text{.............................(1).}$$

The velocity from E to G, parallel to BO, is v; hence

$$b = v\,(t_1 + t_2) \qquad\text{..........................(2).}$$

The velocity from F to H, parallel to OA, is eu; hence

$$x = eu\,(t_2 + t_3) \qquad\text{..........................(3).}$$

The velocity from G to H, parallel to OB, is ev; hence

$$y = ev\,.\,t_3 \qquad\text{.......................... (4).}$$

The whole time of flight is $t_1 + t_2 + t_3$; hence

$$\frac{2w}{g} = t_1 + t_2 + t_3 \qquad\text{........................(5).}$$

From (1), (3), (5), we see that

$$x = e\left(\frac{2uw}{g} - a\right);$$

and, from (2), (4), (5),

$$y = e\left(\frac{2vw}{g} - b\right).$$

These expressions for x and y determine the position of H, the required point.

(4) A ball, thrown from any point in one of the walls of a rectangular room, after striking the three others returns to the point of projection before it falls to the ground: shew that the space due to the velocity of projection is greater than the diagonal of the floor.

(5) A ball, of elasticity e, is projected from the lower edge of one of the walls of a room with a square floor, the length of

each side of the square being a feet, at an inclination θ to the horizon, with a velocity which would, if vertical, just carry it to the ceiling of the room: prove that, after hitting the other three walls in succession, it will return to the point of projection, if

$$\sin 2\theta = \frac{a(1+e)}{2e^2 h}\left\{1 + e^2 - 2\frac{x}{a}(1-e) + \frac{x^2}{a^2}(1-e)^2\right\}^{\frac{1}{2}},$$

h being the height of the room, and x the distance of the point of projection from the wall on which the ball first impinges.

(6) From the foot of a plane, inclined to the horizon at an angle α, a ball is projected with a velocity V along the surface of the plane, in a direction making an angle β with the straight line in which the inclined plane intersects a fixed horizontal plane: determine the path and the time which elapses before the ball reaches the foot of the inclined plane again: also, supposing the ball to be elastic, find the range of its rebound on returning to the horizontal plane, and the position of the plane in which the rebound takes place.

The path on the inclined plane will be a parabola, the distances of its vertex and focus from the foot of the plane being respectively

$$\frac{V^2 \sin^2 \beta}{2g \sin \alpha} \quad \text{and} \quad -\frac{V^2 \cos 2\beta}{2g \sin \alpha},$$

the time of flight on the inclined plane being equal to

$$\frac{2V \sin \beta}{g \sin \alpha}.$$

The inclination of the plane of rebound to the rectilinear foot of the inclined plane will be equal to

$$\tan^{-1}(\cos \alpha \,.\, \tan \beta),$$

and, e being the elasticity of the ball, the range of the rebound will be equal to

$$\frac{2eV^2}{g} \,.\, \sin \alpha \,.\, \sin \beta \,.\, (1 - \sin^2 \alpha \,.\, \sin^2 \beta)^{\frac{1}{2}}.$$

CHAPTER VIII.

COLLISION OF PROJECTILES.

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(1) An elastic sphere is projected from a point in a horizontal plane and, at the moment when it reaches its greatest altitude, collides horizontally with an equal sphere moving in the opposite direction with the same velocity: to determine the elasticity of the spheres in order that the former may return to the point of projection after one rebound on the horizontal plane, the elasticity relative to the impact being the same as that relative to the collision.

Let t be the time of moving from the point of projection to the highest point: the time of moving thence to the horizontal plane will be the same: let t' be the time through the rebound, which we will suppose to terminate at the point of projection. Let e be the elasticity.

Since the horizontal velocity during the last two portions of the motion is e times that during the first part of the motion, it is clear that,

$$t = e\,(t + t').$$

Now the time of flight of a projectile varies as the vertical component of the velocity of projection; hence, the first portion of the motion being a half flight, and the last portion a whole one, we see that $t' = 2et$. Hence

$$e\,(1 + 2e) = 1,$$

and therefore $$e = \frac{1}{2}.$$

(2) A sphere Q is dropped from a height BC upon a smooth horizontal plane AC, and, at the same moment, a sphere P is projected from A so as to strike Q at the instant when P is moving parallel to the plane. To determine the circumstances of P's projection; and to ascertain the subsequent motion, (1) when P impinges directly upon Q, (2) when Q impinges directly upon P; both P and Q being smooth and inelastic.

Let $AC = a$, $BC = b$; and let m, m', be the masses of P, Q, respectively. Let V, α, be the velocity and angle of projection of P, and t the interval between the commencement of the motion and the collision.

Then $$V \cos \alpha \, . \, t = a \quad \text{.......................... (1)},$$

and, $V \sin \alpha$ being the relative vertical velocity of the two spheres and b their initial vertical distance,

$$V \sin \alpha \, . \, t = b \quad \text{.......................... (2)}.$$

Also, P's motion being horizontal the instant before collision,

$$gt = V \sin \alpha \quad \text{.................... (3)}.$$

From (1) and (2), we see that

$$\tan \alpha = \frac{b}{a} \quad \text{.......................... (4)},$$

a result which shews that the sphere P must be projected in the direction AB.

From (2) and (3), we have

$$t = \left(\frac{b}{g}\right)^{\frac{1}{2}} \quad \text{...................... (5)}.$$

From (1), (4), (5), we see that

$$V = a \left(\frac{g}{b}\right)^{\frac{1}{2}} . \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}}$$

$$= \left(\frac{g}{b}\right)^{\frac{1}{2}} . (a^2 + b^2)^{\frac{1}{2}}.$$

The horizontal velocity of P, before collision, being $V \cos \alpha$, the horizontal velocity of each sphere afterwards, supposing P to impinge directly upon Q, will be equal to

$$\frac{m}{m + m'} . V \cos \alpha,$$

and therefore, by (1) and (5), to

$$\frac{ma}{m + m'} . \left(\frac{g}{b}\right)^{\frac{1}{2}}.$$

The vertical velocity of Q, just after as well as just before collision, is equal to

$$gt = (bg)^{\frac{1}{2}}, \text{ by (5)}.$$

Supposing Q to impinge directly upon P, the vertical velocity of each sphere, just after collision, is equal to

$$\frac{m'gt}{m+m'} = \frac{m'(bg)^{\frac{1}{2}}}{m+m'},$$

the horizontal velocity of P being, both before and after the collision, equal to

$$V\cos\alpha = \frac{a}{t}, \text{ by (1)},$$

$$= a\left(\frac{g}{b}\right)^{\frac{1}{2}}, \text{ by (5)}.$$

(3) Two balls, of elasticity e, are projected towards each other in the same vertical plane, at the same instant, from two points in a horizontal line, V being the velocity and α the angle of projection of each: determine the distance between the points of projection in order that, after colliding together, they may return to these points.

The required distance is equal to

$$\frac{2eV^2\sin 2\alpha}{(1+e)g}.$$

(4) Two equal imperfectly elastic balls are projected simultaneously up the sides of an isosceles right-angled triangle, of which the plane is vertical and hypotenuse horizontal, so as just to reach the top: supposing the triangle to be removed, and the balls to be projected with the same velocities and in the same directions as before, shew that, if the focus of the parabola, in which the balls move after collision, occupies the position of the centre of gravity of the triangle, the elasticity of the balls is $\frac{1}{\sqrt{3}}$.

(5) Two equal and perfectly elastic spherical particles P, Q, are projected at the same instant in the same vertical plane from

two points A, B, in the same horizontal line, at a distance $\frac{1}{2}g\sqrt{3}$ from each other; the former vertically upwards with a velocity g, and the latter, at an angle of 30° to BA with a velocity $2g$; determine where they will strike the horizontal line on their fall.

P will strike the horizontal line BA, produced, at a point C, such that $AC = 3AB$, and Q will fall at the point A.

(6) Two balls are thrown from given points A, B, in the same horizontal line, so as to move in one plane, and collide horizontally at their highest points: find the position of the point of collision, their horizontal velocities being given.

If AB be represented by c, and u, u', be the horizontal velocities of the balls, the altitude of the point of collision above AB is equal to

$$\frac{gc^2}{2(u+u')^2},$$

and the distances of this point from verticals through A, B, are as u to u'.

(7) If m, m', be the respective masses of the balls in the preceding problem, and e their mutual elasticity, prove that m will return to A or m' to B, respectively, accordingly as $\frac{2u}{m'}$ or $\frac{2u'}{m}$ is equal to

$$(1+e)\cdot\frac{u+u'}{m+m'}.$$

Prove also that, in any case, if the balls strike AB at points A', B',

$$A'B' = e\cdot AB.$$

(8) A tube of small bore, open at both ends, is bent into the form of two-thirds of the circumference of a circle and is placed in a vertical position with the line joining the open ends horizontal: if two equal balls start from rest from the highest

point, prove that, after leaving the tube, they will collide at a point the distance of which below the open ends is equal to the diameter, and that, if the elasticity be $\frac{1}{3}$, the distance between the directrixes of the parabolas, described before and after collision, is one-sixth of the diameter.

CHAPTER IX.

IMPACT AND COLLISION OF BODIES MOVING UNDER CONSTRAINT.

SECT. 1. *Impact.*

(1) A heavy particle, of elasticity e, is projected with a velocity V obliquely up a given smooth inclined plane, so as just to reach the top, and, at that point, impinges directly on a smooth fixed plane: to find the distance between the point of projection and the point in the same horizontal plane through which the particle passes in its descent.

Let A, fig. (131), be the point of projection, B the point of impact; let AC be the horizontal line through A in the inclined plane, D the point in AC through which the particle passes in its descent: draw BC at right angles to AC: a plane through BC, at right angles to the inclined plane, is the plane on which the particle impinges at B, because, by the nature of the question, the motion of the particle just before its impact, which is direct, is horizontal.

Let $BC = l$, $h =$ the vertical height of B above the horizontal plane through A, $\beta =$ the inclination of the direction of projection to AC, $t =$ the time from A to B. Then, $V\cos\beta$ being the component of the velocity of projection along AC, we have

$$AC = V\cos\beta \, . \, t \text{..........................(1)}.$$

Again, the time of descent from B to D must be equal to that of ascent from A to B, because at B the velocity parallel to BC is zero: hence, $eV\cos\beta$ being the velocity at B after impact,

$$CD = eV\cos\beta \, . \, t \text{..........................(2)}.$$

From (1) and (2) we have

$$AD = (1 - e)\, V\cos\beta \, . \, t \text{.................. (3)}.$$

Again, considering the motion parallel to BC, we have

$$V^2 \sin^2 \beta = 2gh \text{.......................} (4),$$

and
$$t^2 = \frac{2l^2}{gh} \text{.......................} (5).$$

From (3), (4), (5), we see that the required distance AD is equal to

$$(1-e) \, . \, (V^2 - 2gh)^{\frac{1}{2}} \, . \left(\frac{2l^2}{gh}\right)^{\frac{1}{2}}.$$

(2) Two smooth planes, each inclined to the horizon at an angle α, less than $\frac{\pi}{4}$, intersect in a horizontal line: an inelastic ball is projected from a point in the above intersection along one of the planes, in a direction making an angle β with the intersection: to prove that the ball will cross from one plane to the other at intervals which decrease in geometrical progression, and will cease to cross after a time

$$\frac{V \sin \beta}{g \sin^3 \alpha},$$

at a distance from the point of projection equal to

$$\frac{V^2 \sin 2\beta}{2g \sin^3 \alpha}.$$

The components of the velocity of projection, parallel and perpendicular to the line of intersection, are $V \cos \beta$, $V \sin \beta$.

Consider first the component of the motion which is at right angles to the line of intersection.

The interval of time between the instant of projection and the first crossing is equal to

$$\frac{2 V \sin \beta}{g \sin \alpha}.$$

The component of the velocity, at right angles to the line of intersection, with which the ball commences the ascent of the next plane, is equal to

$$V \sin \beta \cos 2\alpha:$$

hence the interval between the first and second crossing is equal to

$$\frac{2V \sin \beta \cos 2\alpha}{g \sin \alpha}:$$

similarly, the ball's component velocity, at the beginning of the third ascent, is equal to $V \sin \beta . \cos^2 2\alpha$, and the interval between the second and third crossing is equal to

$$\frac{2V \sin \beta \cos^2 2\alpha}{g \sin \alpha},$$

and so on indefinitely. Thus the series of component velocities and intervals form two geometrical progressions, the common ratio of each of which is $\cos 2\alpha$, a ratio less than unity: thus the ball's velocity of ascent, at right angles to the line of intersection, will be zero, and it will therefore cease to cross, after a time, dated from the commencement of the motion, equal to

$$\frac{2V \sin \beta}{g \sin \alpha} (1 + \cos 2\alpha + \cos^2 2\alpha + \ldots \text{ ad inf.})$$

$$= \frac{2V \sin \beta}{g \sin \alpha (1 - \cos 2\alpha)} = \frac{V \sin \beta}{g \sin^3 \alpha}.$$

The distance of the ball from the point of projection, at the end of this time, since the component of the velocity, parallel to the line of intersection, is constantly $V \cos \beta$, is equal to

$$\frac{V^2 \sin \beta \cos \beta}{g \sin^3 \alpha} = \frac{V^2 \sin 2\beta}{2g \sin^3 \alpha}.$$

(3) A particle of given elasticity slides down a given inclined plane, and impinges on a horizontal plane at the foot of the given plane: find the range of the particle after reflection.

If e denote the elasticity of the particle, l the length of the inclined plane, and α its inclination, the range is equal to

$$2el \sin \alpha \sin 2\alpha.$$

(4) A particle slides down an inclined plane of given height: if, at the bottom of the plane, it rebounds from a hard

horizontal plane, determine the inclination of the former that the range on the latter may be the greatest possible.

The required inclination is 45°.

(5) If an inelastic particle descends from the highest to the lowest point of a vertical circle by sliding down two chords, prove that it is indifferent, as regards the time occupied, how the chords be taken; prove also that the sum of the squares of the two final velocities will be constant.

(6) *AB* is a smooth inclined plane, terminating in a smooth horizontal plane *BC*: a number of balls slide down different lengths of *AB*, and are reflected by *BC*: prove that, if they all strike *BC* again at the same distance from *B*, their elasticities must vary inversely as the lengths of *AB* severally described by them.

(7) A regular hexagon, made of wire, is placed with one diagonal vertical, and a small ring slides down the circumference from the highest point to the lowest; compare the velocity acquired with that due to falling down the diagonal.

The velocity acquired by falling down the circumference is to that due to falling down the diagonal as 5 to 8.

(8) A ball, of elasticity e, is projected, with a velocity V, directly up a smooth plane inclined to the horizon at an angle α: when the ball has moved through a distance c, it strikes a smooth vertical plane, the intersection of which with the former plane is a horizontal line: prove that, if

$$V^2 = gc\left\{2\sin\alpha + \frac{\operatorname{cosec}\alpha}{2e(1+e)}\right\},$$

it will rebound to the point of projection; and that, V being given, the range along the plane will be a maximum and equal to

$$\frac{e(1+e)V^4}{4g^2c}, \quad \text{if } \sin\alpha = \frac{V^2}{4gc}.$$

(9) A ball is projected with a given velocity along a smooth horizontal plane: it meets a small inclined plane, the intersection of which with the horizontal plane is perpendicular

to the line of the body's motion: find the greatest altitude to which the body afterwards rises and the horizontal range.

If α = the inclination of the small plane, e = the elasticity of the ball, and V = the velocity of projection, the greatest altitude to which the body afterwards rises and the horizontal range are respectively equal to

$$\frac{V^2}{8g}(1+e)^2 \sin^2 2\alpha,$$

and $$\frac{V^2}{g}(1+e) \, . \, (\cos^2\alpha - e\sin^2\alpha) \, . \, \sin 2\alpha.$$

(10) AOB is the vertical diameter of a vertical circle, of which O is the centre: a perfectly elastic particle slides down any chord AC, and, being reflected by the chord BC, describes its path as a projectile: find the locus of the foci of all the parabolas.

The required locus is a circle of which AO is a diameter.

(11) An imperfectly elastic ball slides down a smooth inclined plane, and impinges upon a smooth horizontal plane at rest: find the time which will elapse before the ball ceases to hop.

If c be the length and α the inclination of the inclined plane, then, e denoting the elasticity of the ball, the required time is equal to

$$\frac{2e}{1-e} \cdot \left(\frac{2c}{g}\right)^{\frac{1}{2}} . \, (\sin\alpha)^{\frac{3}{2}}.$$

(12) The axis of a parabola is vertical, the vertex downwards; a circle, the centre C of which is a point in the parabola, passes through the focus S: a perfectly elastic particle, sliding down CS, is reflected at S by the circle, then moves freely under the action of gravity, and finally strikes the circle at its lowest point: find the length of CS.

CS is equal to two-thirds of the latus rectum.

SECT. 2. *Collision.*

(1) Two particles, the common elasticity of which is $\frac{1}{2}$, are let fall in opposite directions at the same instant from the highest

point of a smooth circular tube of very small bore, placed in a vertical position; to find the ratio of their masses, in order that the heavier particle may remain at rest after collision, and to determine the height to which the other will rise.

Let r be the radius of the circle, m' the mass of the larger and m of the smaller particle, u the velocity of each particle the instant before collision, and v the velocity of m just after collision.

Then $$u^2 = 4gr \quad\text{.......................... (1).}$$

Again, the sum of the momenta being the same just before and just after collision,

$$m'u - mu = mv \quad\text{...................... (2),}$$

and, the relative velocity after collision being e times that before collision,

$$e \cdot 2u = v \quad\text{.......................... (3).}$$

From (2) and (3) we see that

$$(m' - m)\,u = 2emu,$$

and therefore

$$m' = (1 + 2e)\,m = \frac{3}{2}m.$$

Also, h being the height to which m will rise after collision,

$$\begin{aligned} 2gh &= v^2 \\ &= 4e^2u^2, \quad \text{by (3),} \\ &= 16e^2gr, \quad \text{by (1),} \\ &= gr, \end{aligned}$$

and therefore $$h = \frac{1}{2}r.$$

(2) AB is an inclined plane: a ball of mass m slides from rest at A down AB, to meet a ball of mass m', which starts at the same time from B directly up the plane: after collision, m just returns to A: to find the starting velocity of m', the place of meeting, and the motion of the centre of gravity of the balls.

Let l be the length and α the inclination of AB to the horizon: let u be the velocity of m' at starting, t the time between the commencement of the motion and the collision: v, v', the velocities

of m, m', the instant before collision, estimated respectively along AB, BA; and v_1 the velocity of m', the instant after collision, estimated along AB. The velocity of m, the instant after collision, must, by the condition of the problem, be v along BA. Let e be the elasticity of the balls.

Then we have

$$v = g \sin \alpha \,.\, t \text{..........................(1)},$$

and

$$v' = u - g \sin \alpha \,.\, t \text{.....................(2)}.$$

Again, since gravity has no effect upon the relative motion of the balls, and since, during the time t they have approached each other by the space l, we see that

$$l = ut \text{..............................(3)}.$$

Again, by the equations of collision, we have

$$m'v' - mv = mv - m'v_1,$$

or

$$m'(v' + v_1) = 2mv \text{.....................(4)},$$

and

$$e(v + v') = v + v_1 \text{.....................(5)}.$$

From (1), (2), (5), we get

$$v + v' = u \text{ and } v + v_1 = eu,$$

and therefore, by (4),

$$m'\{(1+e)u - 2v\} = 2mv,$$

$$(1+e)m'u = 2(m+m')v:$$

but, by (1) and (3),

$$v = \frac{gl \sin \alpha}{u};$$

hence

$$u^2 = \frac{2gl \sin \alpha}{1+e} \cdot \frac{m+m'}{m'}.$$

Again, the distance of the point of collision from A equal to

$$\tfrac{1}{2}g \sin \alpha \,.\, t^2 = \tfrac{1}{2}g \sin \alpha \,.\, \frac{l^2}{u^2}$$

$$= \tfrac{1}{4}(1+e) \,.\, l \,.\, \frac{m'}{m+m'}.$$

Also, the initial velocity of the centre of gravity, in the direction AB, being equal to

$$\frac{m'u}{m+m'},$$

its velocity, at the end of any time t', in the same direction, since it is not affected by collision, may be always represented by the expression

$$\frac{m'u}{m+m'} - gt' \sin \alpha.$$

(3) Two unequal weights, P and Q, are connected by a fine string of given length, which passes over a smooth fixed pully: after P has descended through a feet, it impinges on a fixed inelastic horizontal plane: to find the time which will elapse before the system comes to a state of permanent rest.

Let f = the acceleration of P, Q, when both are moving. Let u_n = the velocity of $P+Q$, the instant before the n^{th} impact on the horizontal plane. The time during which Q will move, subsequently to the impact, before disturbing P, will be equal to $\frac{2u_n}{g}$. At the end of Q's free motion, that is, the moment the string is again tightened, the velocity of $P+Q$ will be equal to

$$\frac{Qu_n}{P+Q} \quad \text{.............................} (1),$$

and accordingly the whole time of P's ascent and descent will be equal to

$$\frac{2Qu_n}{(P+Q)f} = \frac{2Qu_n}{(P-Q)g}.$$

Hence the whole time between the n^{th} and $(n+1)^{\text{th}}$ impacts is equal to

$$\frac{2u_n}{g}\left(1+\frac{Q}{P-Q}\right) = \frac{2Pu_n}{g(P-Q)} \quad \text{..............} (2).$$

Again, since the velocity of $P+Q$ is the same just before the $(n+1)^{\text{th}}$ impact as just after the first tightening of the string after the n^{th} impact, we have, by (1),

$$u_{n+1} = \frac{Qu_n}{P+Q} \quad \text{......................} (3).$$

This relation shews that the velocities $u_1, u_2, u_3, \ldots$ form a geometrical progression of which $\frac{Q}{P+Q}$ is the common ratio; hence $u_n = 0$ when $n = \infty$: thus, when $n = \infty$, the system is permanently at rest.

The whole time therefore which will elapse, after the first impact, before the system comes to permanent rest, is, by (2), equal to

$$\frac{2P}{g(P-Q)}(u_1 + u_2 + u_3 + \ldots \text{ ad inf.})$$

$$= \frac{2Pu_1}{g(P-Q)}\left\{1 + \frac{Q}{P+Q} + \frac{Q^2}{(P+Q)^2} + \ldots \text{ ad inf.}\right), \text{ by (3)},$$

$$= \frac{2Pu_1}{g(P-Q)} \cdot \frac{P+Q}{P}$$

$$= \frac{2(P+Q)}{g(P-Q)} \cdot u_1$$

$$= \frac{2(P+Q)}{g(P-Q)} \cdot \left(2a\frac{P-Q}{P+Q}g\right)^{\frac{1}{2}}$$

$$= 2\left(\frac{2a}{g} \cdot \frac{P+Q}{P-Q}\right)^{\frac{1}{2}}.$$

(4) A cannon in recoiling is made to run up a smooth curve, the lowest element of which is horizontal: the vertical altitude of the highest point to which the cannon runs up this curve being observed, and the weights of the cannon and the ball being given, find the velocity of the ball.

If m, m', be the respective masses of the ball and cannon, and h the vertical ascent of the cannon, the required velocity is equal to $\frac{m'}{m}(2gh)^{\frac{1}{2}}$.

(5) Two weights P, Q, are connected by a fine string passing freely over a smooth fixed pully: if an additional weight $P+Q$, not in motion, be suddenly attached to either weight, at any instant during the motion, determine the sudden change of velocity.

The velocity of the weights is suddenly diminished by one half.

(6) A perfectly elastic ball is projected obliquely, and, on reaching its highest point, strikes directly another equal ball hanging by a string from the directrix of its path: prove that the struck ball will just reach the directrix.

(7) Two given inelastic weights are connected by an inextensible string, which passes over a smooth pully: the system being initially at rest, find the weight which, let fall at the beginning of the motion from a point vertically above the ascending weight, so as to impinge directly upon it, will instantaneously reduce the system to rest. Will the system afterwards remain at rest?

If P be the larger and Q the smaller of the two given weights, the required weight is equal to $P - Q$. The system will remain permanently at rest.

(8) Two masses m, m', are connected by an inextensible string, which is placed over a smooth peg, and the system is abandoned to the action of gravity: at the end of an interval τ, a mass μ, not in motion, is suddenly attached to the smaller of the two masses, and this operation is repeated at the end of each succeeding interval τ: prove that, if

$$\frac{2(m \sim m')}{\mu} = p,$$

where p is an integer, the system will come to rest at the end of the time $(p+1)\tau$.

(9) Two perfectly elastic balls, of equal volume and weight, are suspended by thin rigid wires from two fixed points in the same horizontal line: the length of each wire, measured from the centre of the ball, is a, and, when the wires are vertical, the balls are in contact: one ball is raised and allowed to impinge on the other: determine the motion of the balls after successive impacts; and shew that their centre of gravity describes between successive impacts an arc of a circle, of diameter a, in the time of oscillation of a simple pendulum of length a.

CHAPTER X.

DYNAMICAL UNITS.

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(1) A BODY, moving uniformly, is observed to describe a feet in n seconds: supposing the unit of time to be the $\left(\frac{p}{q}\right)^{\text{th}}$ of a minute, what must be the unit of distance in order that the numerical value of the body's velocity may be unity?

The body describes a feet in n seconds, and therefore $\frac{a}{n}$ feet in 1 second, and therefore $\frac{pa}{qn} \times 60$ feet in the $\left(\frac{p}{q}\right)^{\text{th}}$ of a minute, i.e. in a unit of time; supposing therefore the velocity of the body to be unity, the unit of distance must be equal to $\frac{60\,pa}{qn}$.

(2) What is the numerical value of the acceleration of a falling particle, taking an inch as the unit of space, and half a second as that of time? Employ this numerical measure to find the height which a particle, projected vertically upwards with the velocity of 32.2 feet per second, will rise; and test the correctness of your result by comparing it with that obtained by taking a foot and a second as the units of length and time.

The space which would be described in one second, with the velocity generated in a falling body in one second, is equal to 32.2 feet and therefore to 32.2×12 inches: hence the space which would be described in half a second, with the velocity generated in half a second, is $\frac{1}{4} \times 32.2 \times 12$ inches, that is, 96.6 inches. Hence the numerical value of the acceleration of a falling particle, an inch and a half-second being the respective units of length and time, is 96.6.

If a heavy particle be projected vertically upwards, with a velocity of 32.2 feet per second, it is projected with a velocity of $\frac{1}{2} \times 12 \times 32.2$ inches per half-second: hence, h being the required height of ascent in inches,

$$\left(\frac{1}{2} \times 12 \times 32.2\right)^2 = 2 \times 96.6 \times h,$$

$$6^2 \times 32.2 = 2 \times 3 \times h,$$

$$h = 6 \times 32.2 \text{ inches}:$$

thus the height of ascent in feet is 16.1.

Taking a second as the unit of time and a foot as that of length, we have, h being the required height in feet,

$$(32.2)^2 = 2 \times 32.2 \times h,$$

$$h = 16.1.$$

Thus the two values of h coincide.

(3) The measure of gravity being 32, when a foot is the unit of length and a second that of time; to find the unit of time, when the unit of length is a yard, and 24 is the measure of gravity.

Let t seconds be the required unit of time. Now, the space, which would be described in one second with the velocity generated in one second, being 32 feet, the space which would be described in t seconds, with the velocity generated in t seconds, is equal to $32t^2$ feet: consequently $32t^2$ feet is equal to the measure of gravity, when t seconds are the unit of time. But, by the hypothesis, the measure of gravity, the new unit of time being adopted and the unit of length being a yard, is 24. Hence

$$32t^2 = 24 \times 3, \quad t^2 = \frac{9}{4}, \quad t = \frac{3}{2}.$$

(4) A body describes 75 feet from rest and acquires a velocity of 20, under the action of a uniform force 8, in 5 seconds: what have been taken as units of time and length?

Let t seconds be the unit of time and a feet the unit of space.

Then the space which would be described by the body in t seconds, with the velocity generated in t seconds, is equal to $8a$ feet, and therefore the space which would be described in t seconds, with the velocity generated in one second, is equal to $\frac{8a}{t}$ feet, and the space which would be described in t seconds, with the velocity generated in 5 seconds, is equal to $\frac{40a}{t}$ feet. But, by the hypothesis, this last space is also equal to $20a$ feet: hence

$$\frac{40a}{t} = 20a, \quad t = 2,$$

or the unit of time is equal to 2 seconds.

Again, for the determination of a, we have

$$(20a)^2 = 2 \times 75 \times 8a,$$

and therefore $a = 3$; that is, the unit of length is equal to three feet.

The unit of time may be determined also briefly in the following manner.

Let τ units of time be equal to 5 seconds: then, since a velocity 20 is generated by a uniform force 8 in τ units of time,

$$20 = 8\tau, \quad \tau = \frac{5}{2};$$

thus $\frac{5}{2}$ units of time are equal to 5 seconds, and therefore one unit of time is equal to 2 seconds.

(5) If f be the measure of a uniform acceleration, when t minutes and a feet are taken as the units of time and length, to find the number of minutes in the unit of time, in order that f' may be the measure of the same acceleration, when a' feet are taken as the unit of length.

Let t' denote the required number of minutes in the latter unit of time.

The space which would be described in t minutes, with the velocity added in t minutes, is equal to fa feet, and therefore the

space which would be described in one minute, with the same velocity, is equal to $\frac{fa}{t}$ feet, and consequently the space which would be described in one minute, with the velocity added in one minute, is equal to $\frac{fa}{t^2}$ feet. Hence the space which would be described in t' minutes, with the velocity added in one minute, is equal to $\frac{fa}{t^2}\,.\,t'$ feet, and therefore the space which would be described in t' minutes, with the velocity added in t' minutes, is equal to $\frac{fa}{t^2}\,.\,t'^2$ feet. But $f'a'$ also represents the number of feet which would be described in t' minutes with the velocity added in t' minutes: hence

$$f'a' = fa\,.\,\frac{t'^2}{t^2},$$

and therefore

$$t' = t\left(\frac{f'a'}{fa}\right)^{\frac{1}{2}}.$$

Campion and Walton: *Solutions of the Senate-House Problems and Riders for the year* 1857, p. 85.

(6) If the force of gravity be taken as the unit of force, and a rate of ten miles an hour as the unit of velocity, what must be the units of length and time?

Let a feet be the unit of length and t seconds the unit of time.

Then the force of gravity, being the unit of force, generates in a unit of time a unit of velocity, that is, the space which would be described in t seconds, with the velocity generated in t seconds, is equal to a feet. But the space which would be described in t seconds, with the velocity generated by gravity in t seconds, is equal to $32.2t^2$ feet: hence

$$a = 32.2t^2 \text{..............................}(1).$$

Again, at the rate of ten miles an hour, the space which would be described in t seconds is equal to

$$\frac{10 \times 1760 \times 3}{60 \times 60}\,t \text{ feet} = \frac{44}{3}\,t \text{ feet}:$$

but this space is the measure of the unit of velocity, that is, a feet: hence

$$a = \frac{44}{3} t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we see that

$$a = \frac{(44)^2}{9 \times 32.2}, \quad t = \frac{44}{3 \times 32.2}.$$

Thus the units of length and time are respectively

$$\frac{(44)^2}{9 \times 32.2} \text{ feet,}$$

and $$\frac{44}{3 \times 32.2} \text{ seconds.}$$

Mackenzie and Walton; *Solutions of the Senate-House Problems and Riders for the year* 1854, p. 121.

(7) To determine the dynamical unit of weight, the mass in a cubic foot of water being the unit of mass, and the unit of accelerating force being the force which in a second generates a velocity of one foot per second.

The unit of pressure is the pressure which generates a unit of velocity in a unit of time in a unit of mass. Hence the unit of weight is a weight which generates a unit of velocity in a unit of time in a unit of mass. But, by the formula $W = Mg$, a weight $\frac{W}{g}$ generates in a mass M a unit of velocity in a unit of time: hence, if the mass in a cubic foot of water be the unit of mass, a foot and a second being the respective units of length and time, the unit of weight is equal to the g^{th} part of the weight of a cubic foot of water, that is, to $\frac{1000}{32.2}$ oz.

(8) A uniform force acts in a fixed direction on a mass of 161 pounds, initially at rest, for one second, and, at the end of that time, has produced a velocity of 20 yards a minute: to compare this force with the weight of 1 pound, the accelerating effect of gravity being 32.2 feet.

Since the force generates in the mass in one second a velocity of 20 yards a minute, that is, a velocity of 1 foot per second, it follows that it would generate in a g^{th} part of the mass in one second a velocity of g feet per second; hence the force is equal to the weight of the g^{th} part of the mass, that is, it bears to one pound the ratio of 161 to 32.2, and therefore of 5 to 1.

(9) State the convention with respect to units which is necessary, in order that the equation $P = Mf$ may represent the relation between the numerical measures of force, mass, and acceleration; and, supposing the unit of force to be 5 lbs., and the unit of acceleration, referred to a foot and a second as units, to be 3, find the unit of mass.

It appears, as the result of experimental facts, that $P \propto Mf$, and therefore that $P = CMf$, the constant C depending on the units assumed. The equation $P = Mf$ implies that the unit of mass is the mass of a body in which the unit of force produces the unit of acceleration, that is, two of the units being given, the assumption that $C = 1$ defines the third.

Let m measure the mass of a body the weight of which is 5 lbs. Then, since a force of 5 lbs. produces in m an acceleration g, where g, referred to a foot and a second as units, is 32.2 approximately, and since, when P is given, $f \propto \dfrac{1}{m}$, it would produce an acceleration '1' in a mass gm, and therefore an acceleration 3 in a mass $\dfrac{1}{3} gm$.

Hence the unit of mass required is to m as g is to 3, and is therefore the mass of a body the weight of which is

$$\frac{32.2}{3} \times 5 \text{ lbs.} = \frac{161}{3} \text{ lbs.}$$

Taking 1000 oz. as the weight of a cubic foot of water, the volume of water representing the unit of mass will be

$$\frac{161}{3} \times \frac{16}{1000} \text{ or } \frac{322}{375} \text{th of a cubic foot.}$$

Campion and Walton: *Solutions of the Senate-House Problems and Riders for the year* 1857, p. 86.

(10) To find the statical measure of the force which in half a mile would stop a railway train of 120 tons weight, moving at the rate of 25 miles per hour.

Let f be the measure of the retardation, a mile being the unit of length and an hour the unit of time: then

$$(25)^2 = 2f \times \frac{1}{2} = f.$$

Hence, the pressure would communicate to the train in one hour, if disturbed from rest, a velocity of $(25)^2$ miles per hour, and therefore in one second a velocity per second of

$$\frac{(25)^2}{(60)^4} \times 1760 \times 3 \text{ feet}$$

$$= \frac{5^4 \times 2^5 \times 5 \times 11 \times 3}{2^8 \times 3^4 \times 5^4} \text{ feet}$$

$$= \frac{5 \times 11}{2^3 \times 3^3} \text{ feet.}$$

Let P denote the required pressure, in tons. Then a force of P tons would generate in one second a velocity of $\frac{5 \times 11}{2^3 \times 3^3}$ feet per second in a mass of 120 tons; and therefore a force of $P \times g \times \frac{2^3 \times 3^3}{5 \times 11}$ tons would generate in one second in the same mass a velocity of g feet per second. Hence, g denoting 32.2,

$$P \times 32.2 \times \frac{2^3 \times 3^3}{5 \times 11} = 120 \text{ tons},$$

$$P = \frac{5^2 \times 11}{3^2 \times 32.2} \text{ tons} = \frac{1375}{1449} \text{th of a ton.}$$

(11) Given that g is the measure of gravity, 1″ being taken as the unit of time, find its measure when the unit is 2′.

The required measure is $14400g$.

(12) If the numerical value of the accelerating force of gravity be g, when a second is taken for the unit of time, what is its numerical value when half a second is taken for the unit?

The required numerical value is $\frac{1}{4}g$.

(13) If 32 be the measure of gravity when a foot and a second are the units of length and time, find the measure when a yard, and ten seconds are chosen for the units.

The required measure is $1066\frac{2}{3}$.

(14) If 32 be the measure of the accelerating force of gravity, find what the unit of time must be in order that the measure may be changed to 128.

The unit of time must be two seconds.

(15) If a line 10 feet long be taken as the unit of length, how many seconds must be taken as the unit of time, that the measure of the accelerating effect of gravity may be 40; 32 being taken as its measure when a foot and a second are taken as the units of length and time?

The required number of seconds is $\frac{5}{\sqrt{2}}$.

(16) If the unit of time be altered in the ratio of 1 to n, and the unit of length in the ratio of 1 to m; in what ratio is the measure of accelerating force altered?

The required ratio is that of n^2 to m.

(17) If the accelerating effect of gravity be taken as the unit of accelerating effect, and a velocity of a mile a minute as the unit of velocity; find the number of seconds and feet in the units of time and length.

The required number of seconds and feet are, approximately, 2.73 and 240.49 respectively.

(18) The accelerating force of gravity being measured by 32.2, when a foot is the unit of length, and a second the unit of time; what will be its measure, when a yard is the unit of length, and two seconds the unit of time?

The required measure will be, approximately, 42.9.

(19) If f, f', be the measures of the accelerating effect of a force when $m+n$ seconds and $m-n$ seconds are the respective units of time, a feet and a' feet being the respective units of

length, find the measure of the accelerating effect of the force when $2m$ seconds are the unit of time, and c feet the unit of length.

The required measure is equal to

$$\frac{1}{c}\{(af)^{\frac{1}{2}} + (a'f')^{\frac{1}{2}}\}^2.$$

(20) What must be the unit of weight, when the weight of a body, the mass of which is the unit of mass, is 3 lbs., a foot and a second being the respective units of length and time?

The required unit of weight is the $\frac{15}{161}$ th part of a pound.

(21) The unit of pressure being 1 lb., and the unit of accelerating force being the force which in a second generates a velocity of one foot per second, what is the unit of mass?

The unit of mass is equal to that contained in a weight of $32\frac{1}{5}$ lbs.

(22) Prove that, under the assumptions by which the unit of mass is determined, its magnitude varies as $\frac{t^2}{s}$, where t seconds and s feet are respectively the units of time and length.

(23) A body, weighing n pounds, is moved by a constant force, which generates in a second a velocity of m feet per second: what weight would the force support?

The required weight is equal to $\frac{mn}{g}$ pounds, where $g = 32.2$.

(24) If the mass of a cubic foot of distilled water, the weight of which is 1000 oz., be taken as the unit of mass, what is the weight, in pounds, of the unit of weight, one foot and one minute being the units of length and time?

The unit of weight is equal to

$$\frac{5^2}{2^6 \times 3^2 \times 161} \text{ pounds.}$$

(25) A uniform pressure of statical intensity equal to 20lbs., will produce, in a body of 3lbs. weight, a velocity of 3 units in one second of time. Assuming $\frac{1}{3}$ lbs. to be the unit of weight, find, (1), the number of feet in the above units, (2), the mass of the body of 3lbs. weight, (3), the weight of the unit of mass: the accelerating force of gravity being assumed to be 32.2 feet.

The unit of length $= 71\frac{5}{9}$ feet, the mass of the body $= 20$ units of mass, and the weight of the unit of mass $= \dfrac{3}{20}$ lbs.

(26) An engine starts a train with a pressure, which continues uniform for five minutes, when it is found that the train is moving at the rate of twenty-three miles an hour: supposing the pressure to remain uniform for fifteen minutes, find the velocity of the train, and, assuming that g is equal to 32.2 feet, find the ratio of the pressure to the weight of the train.

The required velocity of the train is 69 miles an hour and the required ratio is 11 to 3150.

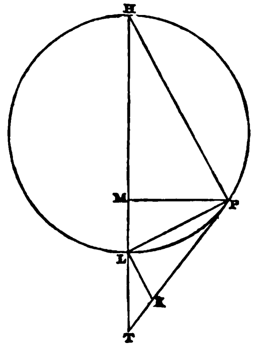
APPENDIX.

THE following very ingenious and refined demonstration of the tautochronism of the cycloid was communicated to me more than a year ago by Mr R. L. Ellis, of Trinity College. He has now, at my request, permitted me to publish it in the present volume. The peculiarity of the demonstration consists in its not introducing the consideration of any secondary property of the cycloid, involving merely the geometrical conception which belongs to the definition of cycloidal generation.

"Conceive two points, not acted on by gravity, to move on the circumference of a stationary circle towards its lowest point: the plane of the circle we will suppose to be vertical. Let their motion be such that the ratio between their distances from the lowest point may be invariable. Then their velocities towards that point must be in that invariable ratio. Their heights above it are in the duplicate ratio[1]: their initial heights above it were also in the duplicate ratio: so likewise therefore are their vertical descents towards it. In other words, the squares of their velocities towards the lowest point are as their vertical descents.

"Conceive one of the points to be, when the other starts, at the highest point of the circle, and to move with a constant velocity $(ga)^{\frac{1}{2}}$, a being the radius of the circle. Then, when it has descended

[1] For let P be one of the points, H the highest and L the lowest point of the circle: join PH, and let PM be at right angles to HL. Then, by the similar triangles LMP, HPL, ML is to PL as PL is to HL, and therefore ML varies as the square of PL.



through a vertical space z from its initial position[1], the square of its velocity towards the lowest point of the circle is equal to $\frac{1}{2}gz$. On this supposition with respect to one point, it appears, from what has been said before, that the square of the velocity of the other point towards the lowest point of the circle is similarly equal to $\frac{1}{2}gz'$, z' being the quantity corresponding to z, viz. its vertical descent below its initial position.

"Now suppose the circle to move horizontally in its own plane with a velocity equal at every instant to the velocity, along the arc, of one of the points, the direction of the motion of the circle being towards the right or the left accordingly as the point is to the right or the left of the vertical diameter. If the former point be chosen, then the velocity of the circle will be constant, if the latter point, it will be variable. In either case, the path of the point selected obviously becomes a cycloid, and it is easily seen that the velocity of the point towards the lowest point of the circle is destroyed by the motion of the circle itself, while the velocity at right angles to this direction is doubled[2]: consequently the whole

---

[1] For let P be the point of which the velocity is $(ga)^{\frac{1}{2}}$. Let the tangent at P meet HL, produced, in T. Join PL and draw LK at right angles to LP. Let V denote the velocity of P in the direction PL. Then

$$V : (ga)^{\frac{1}{2}} :: PL : PK.$$

But the angle KPL is equal to the angle PHM in the alternate segment of the circle, and therefore the triangles KPL, MHP, are similar: hence

$$PL : PK :: HM : HP,$$

and therefore

$$V : (ga)^{\frac{1}{2}} :: HM : HP,$$

or

$$V^2 : ga :: HM^2 : HP^2;$$

but, by the similar triangles HPM, HPL,

$$HM : HP :: HP : HL;$$

hence

$$V^2 : ga :: HM : HL$$
$$:: z : 2a,$$

whence

$$V^2 = \tfrac{1}{2}gz.$$

[2] The angle LPT is equal to the angle LHP in the alternate segment of the circle, and therefore to the angle MPL, because the triangles LHP, PLM, are similar. Hence, the velocity of P in the direction MP, due to the motion of the circle, being equal to P's velocity in the direction PT, due to P's motion in the circle, it follows that P has no motion parallel to PL, and that its motion,

velocity of the point will have, for its square, $2gz$ or $2gz'$, and we have thus a perfect representation of cycloidal motion under the action of gravity. But it is obvious from the fundamental hypothesis that the two points will reach the lowest point of the circle at the same time: that is to say, the descent to the lowest point of the cycloid is tautochronous.

"The analogy to the descent to the lowest point of a circle along its chords is in the essential point complete, but here the motion is not along the chord but along the arc, and is complicated with the motion of the circle itself. In both cases it is easily seen that a medium, the resistance of which varies as the velocity, does not affect the tautochronism."

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at right angles to PL, is double of what it was before the circle had motion: that is, the circle being in motion, the whole velocity of P is double of its velocity towards L while the circle was stationary.

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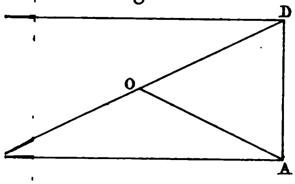
Fig. 3.



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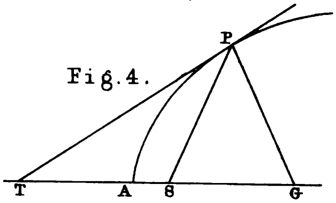

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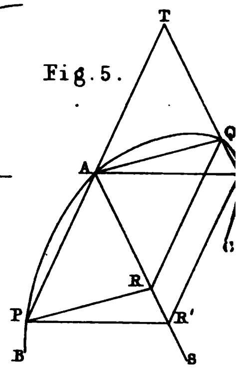

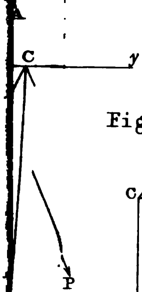

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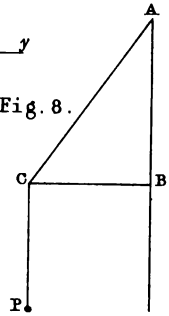

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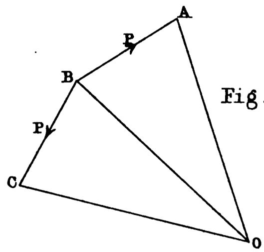

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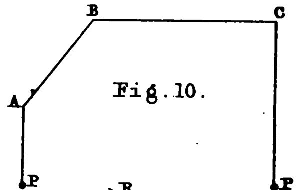

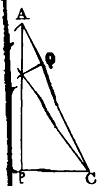

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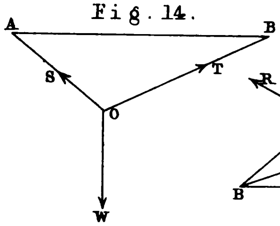

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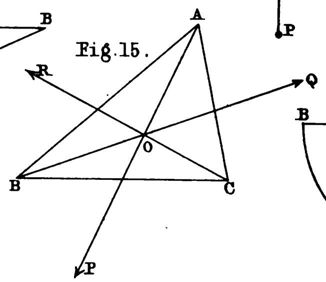

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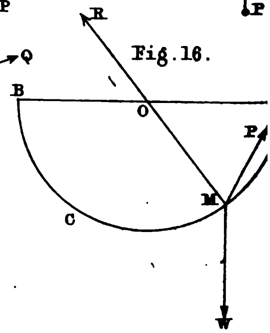

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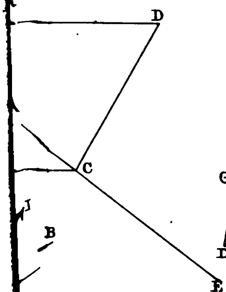

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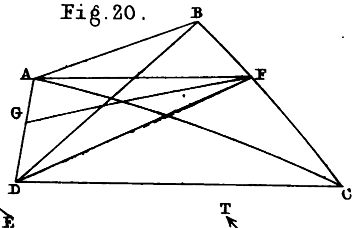

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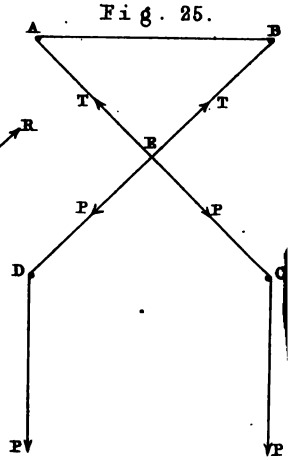

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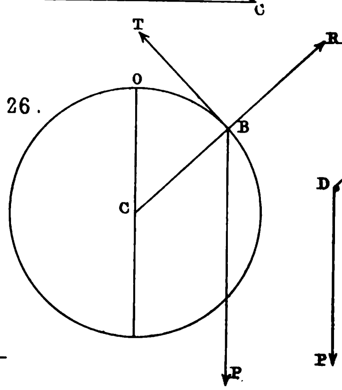

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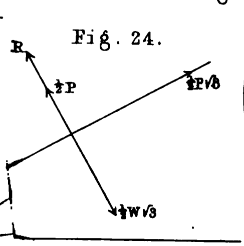

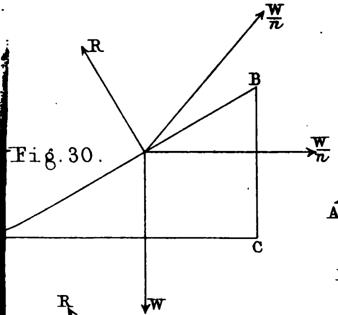

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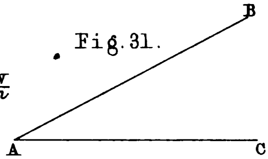

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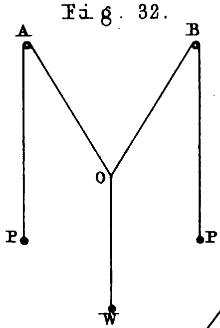

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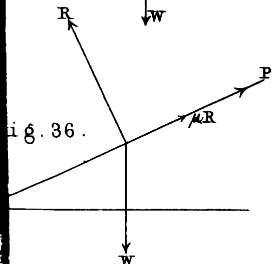

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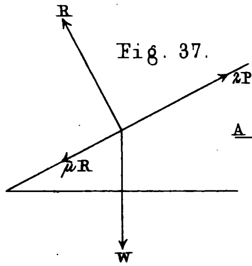

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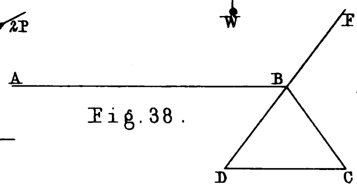

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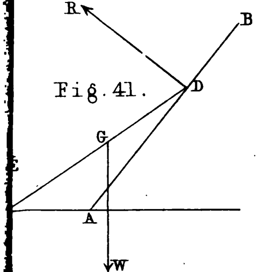

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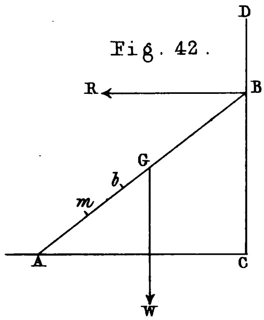

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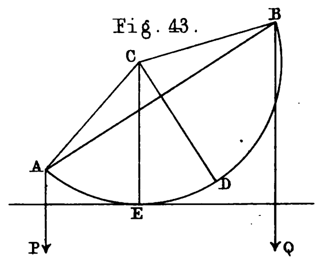

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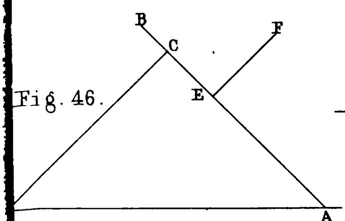

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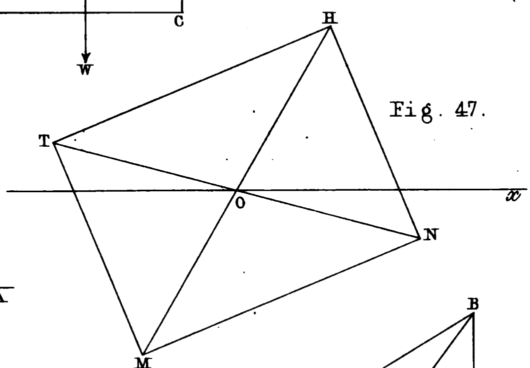

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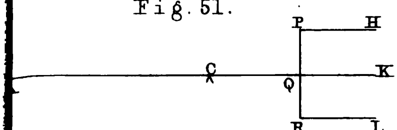

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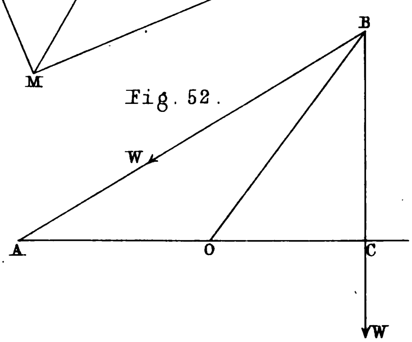

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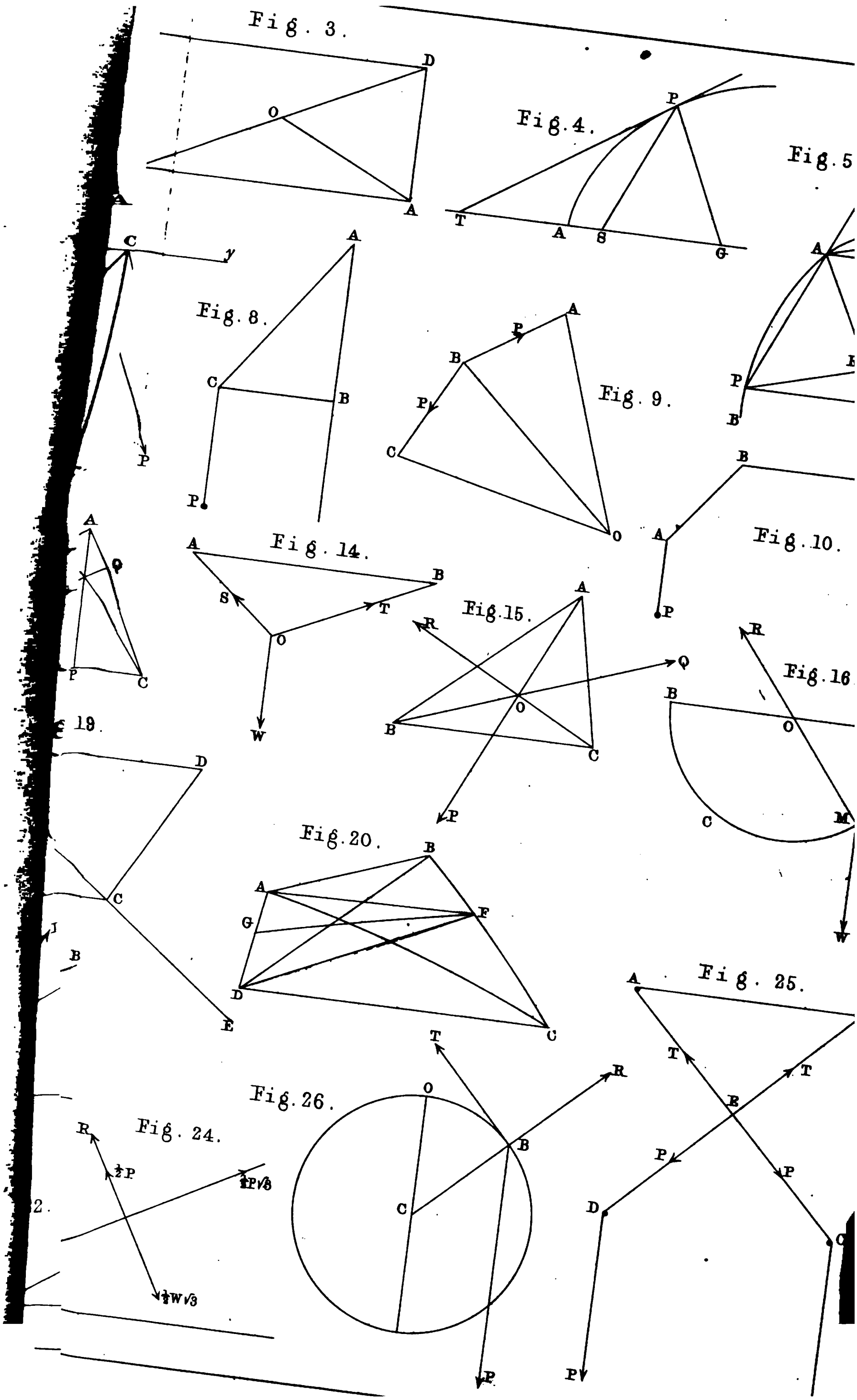

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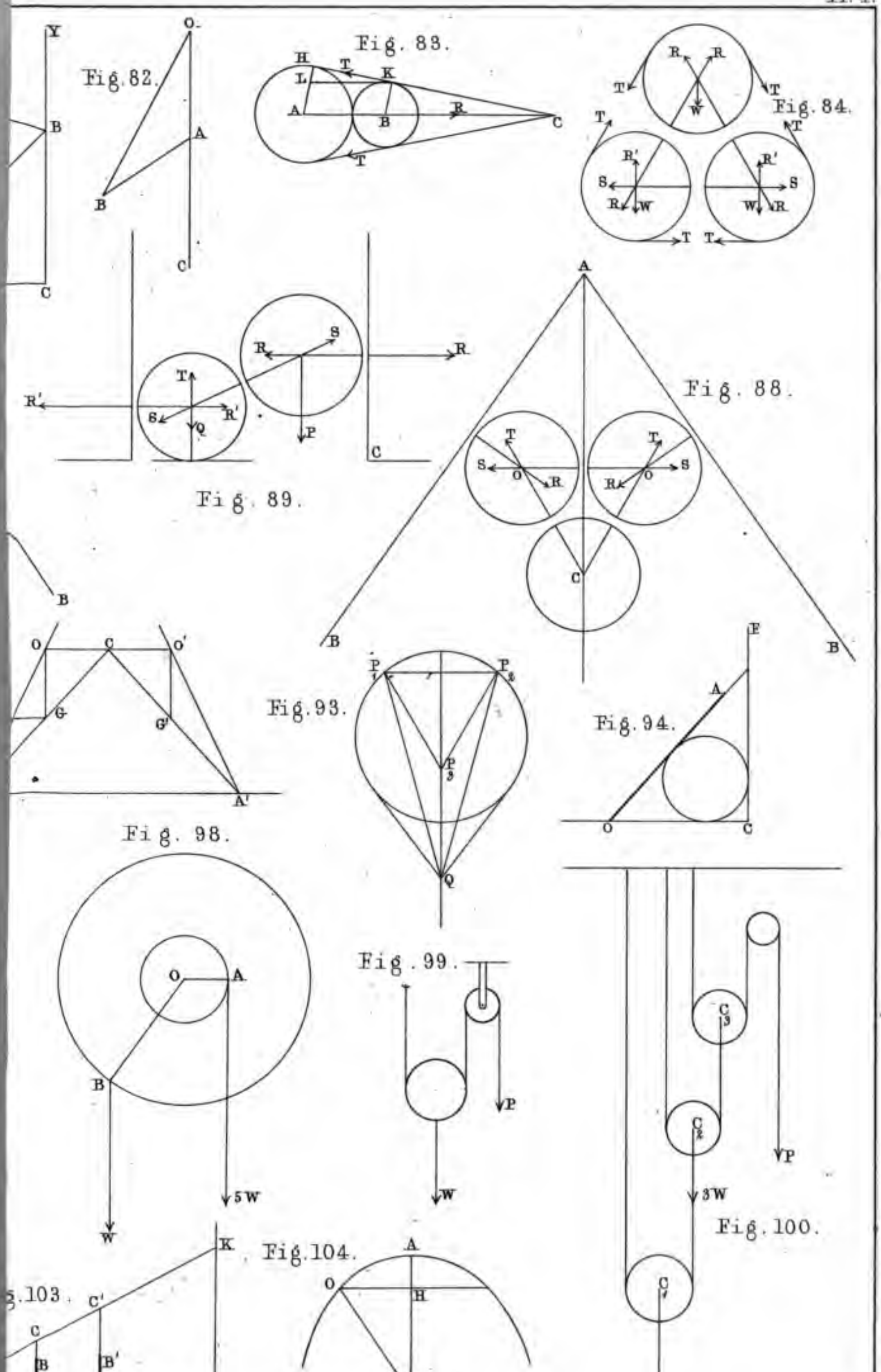

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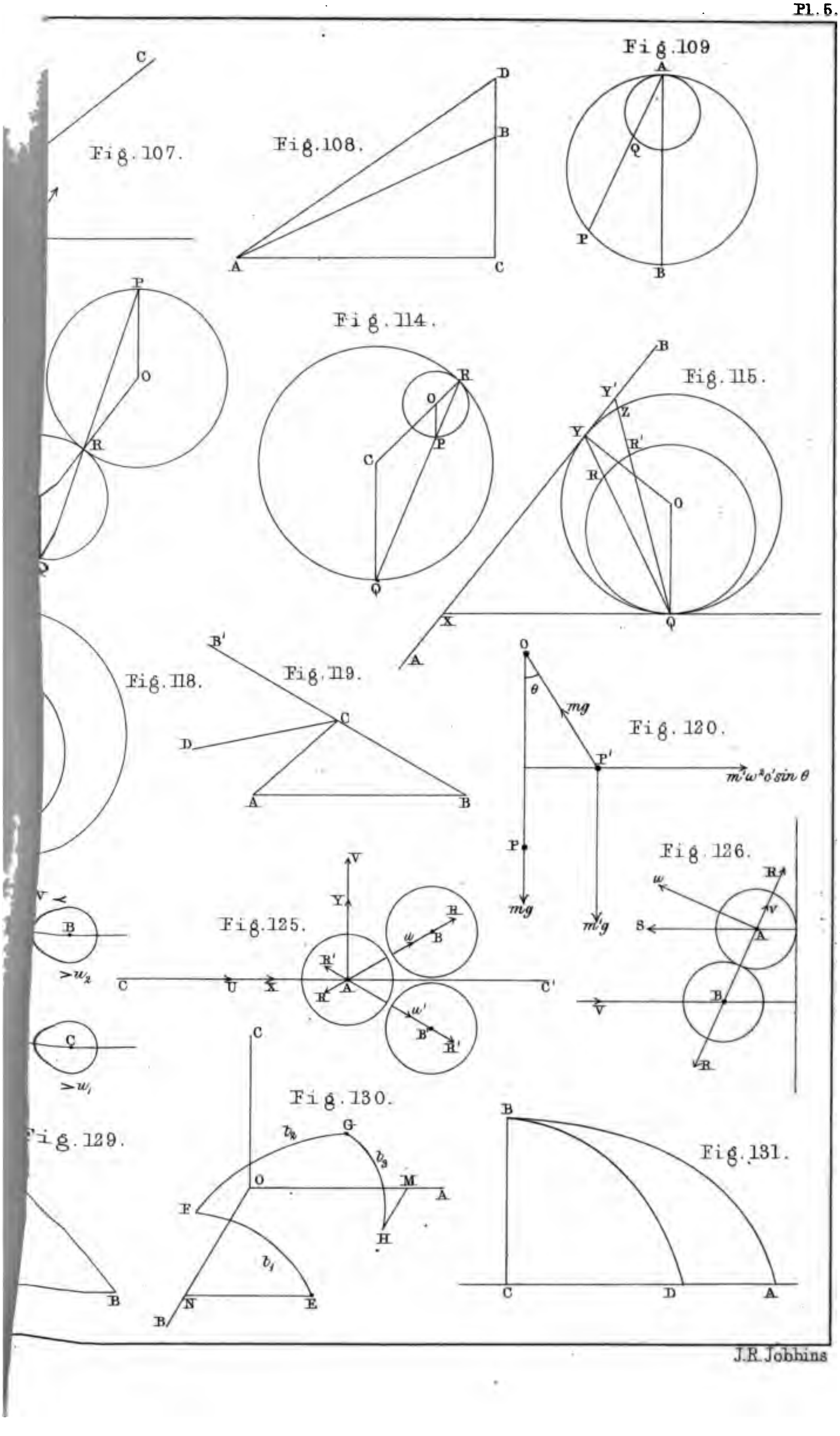
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